Konhauser Problemfest 2017

**1. Limited possibilities?**

Does there exist a function $g$, defined for all real numbers $x$, such that

$$\lim_{x\to 0} g(x) = 0 \quad \text{while} \quad \lim_{x\to 0} g(g(x)) \ \text{exists, but is not zero?}$$

If so, give an example of such a function; if not, prove that there is no such function.

**2. Determine this, please**

Given any positive integer $n$, find the largest integer $k$ for which there exists an $n \times n$ matrix $A$ with nonzero determinant such that $k$ of the entries of $A$ are equal.

**3. A round number?**

For a regular tetrahedron, consider two spheres: the inscribed sphere $S_i$ (which is tangent to each of the four faces) and the circumscribed sphere $S_c$ (which has each of the four vertices on it). Find the ratio $r_i/r_c$ of the radii of these spheres.

**4. A lawsuit waiting to happen**

Alice and Bob were a married couple, whose only child together is Carol. However, Carol is Alice's second child; she had a son, Dave, from a previous marriage. (Bob had no other children.) By law, if a married parent dies without leaving a will, his or her estate gets split equally among his/her spouse and each of his/her children. (That is, the spouse gets one share, and each child gets one share, with all the shares of equal size.) Alas, Alice and Bob died simultaneously, as far as anyone can tell, in an accident, and neither of them left a will. What fraction of their joint estate should Carol inherit?

**5. Let's brighten up the integers!**

Let $n \geq 2$, $k \geq 2$ be integers. Someone had the bright(?) idea of coloring *all* the integers, using at most $k$ given colors, such that both the following rules are satisfied:

- No two consecutive integers can have the same color.
- Any two integers whose difference is $n$ must have the same color.

As a function of $n$ and $k$, in how many different ways can this be done? (Two colorings are considered different when they look different, that is, when at least one integer has different colors under the two colorings.)

**6. A Konhauser anniversary problem**

Evaluate the improper integral

$$\int_0^\infty \frac{25\sin(2017x) - 2017\sin(25x)}{x^2}\, dx\,.$$

(Hint: The answer is not 0.)

(Problems 7 through 10 over $\Longrightarrow$)

## 7. Saving energy

An eccentric inventor has constructed a row of $N$ light bulbs. Each light bulb has a button switch, but pressing the button does several things: It changes the state of that bulb (from on to off or vice versa) and it also changes the state(s) of that bulb's immediate neighbor(s). (Most bulbs have two neighbors, but the ones at the end of the row have only one.)

a) Are there any values of $N > 1$ for which all the lights can be turned off regardless of the initial configuration (which lights are on and which are off)? If so, what are those values of $N$?
b) For any values of $N$ not covered by part a), find the percentage of possible initial configurations for which all the lights can be turned off, and describe which configurations those are.

## 8. Lucky power of a matrix

Find a $3 \times 3$ matrix $A$ with entries from the integers modulo 2, such that $A^7 = I$ but $A \neq I$. (As usual, $I$ denotes the identity matrix, this time with entries modulo 2.)

## 9. A triggy limit of logs?

For values of $x$ sufficiently close to 0, the functions cot and csc can be expressed as

$$\cot x = \sum_{n=0}^{\infty} a_n x^{2n-1}, \quad \csc x = \sum_{n=0}^{\infty} b_n x^{2n-1}.$$

$$\text{(The coefficients start off } a_0 = 1, a_1 = -\frac{1}{3}, a_2 = -\frac{1}{45}, a_3 = -\frac{2}{945}, \ldots$$
$$\text{and } b_0 = 1, b_1 = \frac{1}{6}, b_2 = \frac{7}{360}, b_3 = \frac{31}{15120}, \ldots .)$$

Evaluate (as usual, with proof) the limit

$$\lim_{n\to\infty} \frac{\ln(1 + \frac{b_n}{a_n})}{n}.$$

## 10. A dense problem in the reals?

Do there exist uncountably many subsets of $\mathbb{R}$ such that no two of the subsets have an element in common, each of the subsets is uncountable, and each of the subsets is dense in $\mathbb{R}$? (You may use, without proof, the following lemma, which you may well have seen in the special cases $A = \mathbb{Q}$ and $A = \mathbb{R}$: For any infinite set $A$, the Cartesian product $A \times A$ is equinumerous to $A$.)

Konhauser Problemfest 2017 - solutions
(by Alfonso Gracia-Saz and Mark Krusemeyer)

**1**. Does there exist a function $g$, defined for all real numbers $x$, such that

$$\lim_{x\to 0} g(x) = 0 \quad \text{while} \quad \lim_{x\to 0} g(g(x)) \;\text{ exists, but is not zero?}$$

If so, give an example of such a function; if not, prove that there is no such function.

**Solution**. Yes, there are many such functions; a simple example is given by

$$g(x) = 0 \;\text{ for } x \neq 0\,,\; g(0) = 1\,.$$

For this function, $g(g(x)) = g(0) = 1$ whenever $x \neq 0$, so

$$\lim_{x\to 0} g(g(x)) = 1\,, \quad \text{and} \quad \lim_{x\to 0} g(x) = 0\,.$$

**2**. Given any positive integer $n$, find the largest integer $k$ for which there exists an $n \times n$ matrix $A$ with nonzero determinant such that $k$ of the entries of $A$ are equal.

**Answer**. $k = n^2 - n + 1$.

**Solution**. Let $a$ be the number among the entries of $A$ that occurs most often. For $\det(A)$ to be nonzero, no two of the rows of $A$ can be equal, so at most one row of $A$ can have every entry equal to $a$, while the other $n-1$ rows of $A$ can each have at most $n-1$ entries equal to $a$. Thus there can be at most $n + (n-1)(n-1) = n^2 - n + 1$ equal entries in an $n \times n$ matrix $A$ with nonzero determinant.

To show that it is possible to get this number of equal entries, consider the matrix

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & 1 & \dots & 0 & 1 \\ & & & \dots & & \\ 1 & 1 & 0 & \dots & 1 & 1 \\ 1 & 0 & 1 & \dots & 1 & 1 \end{pmatrix}.$$

If we subtract the first row of this matrix from each of the others, which doesn't affect the determinant, we get

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & -1 \\ 0 & 0 & 0 & \dots & -1 & 0 \\ & & & \dots & & \\ 0 & 0 & -1 & \dots & 0 & 0 \\ 0 & -1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

But it can be seen immediately (by repeated Laplace expansion along successive columns, for example) that the determinant of this last matrix is $\pm 1$, so the original matrix also has nonzero determinant; all but $n-1$ of its entries are 1, so those $n^2 - n + 1$ entries are all equal, as desired.

**3**. For a regular tetrahedron, consider two spheres: the inscribed sphere $S_i$ (which is tangent to each of the four faces) and the circumscribed sphere $S_c$ (which has each of the four vertices on it). Find the ratio $r_i/r_c$ of the radii of these spheres.

**Answer**. $\dfrac{r_i}{r_c} = \dfrac{1}{3}$.

**Solution 1**. Choose coordinates in $\mathbb{R}^3$ so that one vertex of the tetrahedron is the point $(0, 0, 1)$ and the other three vertices are in the $x, y$-plane. The center of both spheres (call it $P$) is the same point, which must be the point $(0, 0, r)$ for some $0 < r < 1$. Then $r_i = r$ and $r_c = 1 - r_i$.

Notice that $P$ is also the center of mass (centroid) of the tetrahedron. By symmetry, it is also the center of mass of the set of four vertices. Thus, each coordinate of $P$ is the average of the corresponding coordinates of the vertices. In particular, the $z$-coordinate is

$$r = \frac{1+0+0+0}{4} = \frac{1}{4}, \quad \text{and it follows that} \quad \frac{r_i}{r_c} = \frac{r}{1-r} = \frac{1/4}{3/4} = \frac{1}{3}.$$

**Solution 2**. Place one face of the tetrahedron in the $x, y$-plane with the origin at the center; without loss of generality, assume the vertices of this face are at

$$(1, 0, 0),\ (-1/2, \sqrt{3}/2, 0),\ (-1/2, -\sqrt{3}/2, 0),$$

so the length of each edge is $\sqrt{3}$. (If you like, you can identify the $x, y$-plane with the complex plane, and the three vertices in this plane with the complex cube roots of unity.) Then by symmetry, the fourth vertex of the tetrahedron is on the $z$-axis, say at $(0, 0, H)$ with $H > 0$. The edge from $(1, 0, 0)$ to $(0, 0, H)$ has the same length $\sqrt{3}$ as the edges in the $x, y$-plane, so $\sqrt{1 + H^2} = \sqrt{3}$, and we see that $H = \sqrt{2}$.

Now by the symmetry of the tetrahedron, the center of both spheres is the same point (it's the only point that is fixed by all the rotations of the tetrahedron), which is a point $(0, 0, h)$ on the $z$-axis. The "inradius" $r_i$ is the distance from this point to the $x, y$-plane (because one face is in that plane), so it is $h$, while the "circumradius" $r_c$ is the distance to any vertex, so it is $H - h$ but also $\sqrt{1 + h^2}$. Using $H = \sqrt{2}$, we see that

$$\sqrt{2} - h = \sqrt{1 + h^2}.$$

Solving this for $h$ yields $h = \dfrac{1}{2\sqrt{2}}$, and we then have

$$\frac{r_i}{r_c} = \frac{1/(2\sqrt{2})}{\sqrt{2} - 1/(2\sqrt{2})} = \frac{1}{4-1} = \frac{1}{3}, \quad \text{as claimed.}$$

**4**. Alice and Bob were a married couple, whose only child together is Carol. However, Carol is Alice's second child; she had a son, Dave, from a previous marriage. (Bob had no other children.) By law, if a married parent dies without leaving a will, his or her estate gets split equally among his/her spouse and each of his/her children. (That is, the spouse gets one share, and each child gets one share, with all the shares of equal size.) Alas, Alice and Bob died simultaneously, as far as anyone can tell, in an accident, and neither of them left a will. What fraction of their joint estate should Carol inherit?

**Answer**. $\dfrac{7}{10}$ .

**Solution**. Let $a$ be the fraction of Alice's estate that Carol should inherit, and $b$ be the fraction of Bob's estate that Carol should inherit. By the law quoted, Alice's estate is split equally between Bob (actually Bob's estate because Bob died at the same instant), Carol, and Dave, so Carol inherits $1/3$ of Alice's estate directly as well as $b/3$ through Bob. That is, $a = 1/3 + b/3$. Similarly, Bob's estate is split equally between Alice (actually Alice's estate) and Carol, so Carol gets half of Bob's estate directly and another $a/2$ through Alice, for a total of $b = 1/2 + a/2$. Solving the simultaneous equations for $a$ and $b$, we find $a = 3/5$, $b = 4/5$. Because the joint estate belonged half to Alice and half to Bob, Carol should inherit $(3/5)(1/2) + (4/5)(1/2) = 7/10$ of the joint estate.

**5**. Let $n \geq 2, k \geq 2$ be integers. Someone had the bright(?) idea of coloring *all* the integers, using at most $k$ given colors, such that both the following rules are satisfied:

- No two consecutive integers can have the same color.
- Any two integers whose difference is $n$ must have the same color.

As a function of $n$ and $k$, in how many different ways can this be done?

**Answer**. $(k-1)^n + (-1)^n(k-1)$ .

**Solution 1**. (This solution was found jointly with Kevin Carde.) If $n = 2$, all the even integers must have one color and all the odd integers must have a different color, so this can be done in $k(k-1)$ ways. Now fix $n \geq 3$ and $k$. A valid way to color the integers is equivalent to a way to color the vertices of an $n$-gon with $k$ colors so that no two adjacent vertices have the same color.

Consider all possible ways to color the vertices of an $n$-gon with $k$ colors (allowing adjacent vertices to have the same color). There are $k^n$ such colorings. For each coloring, call an edge *happy* if the two vertices it joins have different colors, and *sad* if they have the same color. Then we want to count the number of colorings of an $n$-gon with only happy edges; we will do so by subtracting the number of colorings with at least one sad edge from the total number $k^n$ of colorings.

Number the edges (in order, if you like, but actually the choice of numbering doesn't matter) with the integers $1, \ldots, n$, and let $C_i$ be the set of colorings for which edge $i$ is sad. Denote the size of any set $X$ by $|X|$. Then the number of colorings with at least one sad edge is $|C_1 \cup C_2 \cup \cdots \cup C_n|$, and by the inclusion-exclusion principle we have

$$\begin{aligned}|C_1 \cup C_2 \cup \cdots \cup C_n| =& |C_1| + |C_2| + \cdots + |C_n| \\ & -|C_1 \cap C_2| - |C_1 \cap C_3| - \cdots - |C_{n-1} \cap C_n| \\ & +|C_1 \cap C_2 \cap C_3| + \ldots \\ & +(-1)^{n-1}|C_1 \cap C_2 \cap \cdots C_n| \,.\end{aligned}$$

Note that for each $i$, to choose a coloring in $C_i$ we have to specify colors for $n-1$ vertices (then the sadness of edge $i$ gives us the color for the remaining vertex), and so $|C_i| = k^{n-1}$. Similarly, if $i \neq j$, then to choose a coloring in $C_i \cap C_j$ we have to specify colors for $n-2$ vertices, and so $|C_i \cap C_j| = k^{n-2}$. In general, for an $r$-fold intersection $C_{i_1} \cap \cdots \cap C_{i_r}$ with $r \leq n-1$, each of the $r$ sad edges cuts down the number of colors to be chosen independently by 1, so $|C_{i_1} \cap \cdots \cap C_{i_r}| = k^{n-r}$. However, the $n$-fold intersection breaks this pattern, because once $n-1$ edges are sad, the final edge is automatically sad also, and we still have $k$ colors available (to color all vertices identically), so $|C_1 \cap \cdots \cap C_n| = k$.

Note that for any $r$, there are $\binom{n}{r}$ $r$-fold intersections. Thus our "inclusion-exclusion equation" above becomes

$$|C_1 \cup C_2 \cup \cdots \cup C_n| = \binom{n}{1}k^{n-1} - \binom{n}{2}k^{n-2} \cdots + (-1)^n\binom{n}{n-1}k + (-1)^{n-1}k\,,$$

and the desired number of colorings with only happy edges is

$$k^n - |C_1 \cup C_2 \cup \cdots \cup C_n| = k^n - \binom{n}{1}k^{n-1} + \binom{n}{2}k^{n-2} \ldots + (-1)^{n-1}\binom{n}{n-1}k + (-1)^n k.$$

This is the same as the binomial expansion for $(k-1)^n$ *except* for the last term, which in that expansion would be $(-1)^n$ instead of $(-1)^n k$. Therefore, our answer is $(k-1)^n + (-1)^n(k-1)$, as claimed. (Note that this answer is still correct for $n = 2 : (k-1)^2 + (-1)^2(k-1) = k(k-1)$.)

**Solution 2**. Note that because of the second condition, an allowable coloring is completely determined by the colors of the specific integers $1, 2, \ldots, n$. The first condition says 1) that for $1 \leq i \leq n-1$, the integers $i$ and $i+1$ cannot have the same color, and 2) that 1 and $n$ cannot have the same color (else the consecutive integers 0 and 1 would have the same color). Conversely, any coloring of $\{1, 2, \ldots, n\}$ that satisfies 1) and 2) will determine a unique allowable coloring of all the integers.

Now think of $k$ as fixed and let $a_n$ be the desired answer, so $a_n$ is the number of colorings of $\{1, 2, \ldots, n\}$ with at most $k$ given colors that satisfy both conditions 1) and 2). Also, let $b_n$ be the number of colorings of $\{1, 2, \ldots, n\}$ that satisfy 1) but *don't* satisfy 2). We can find a recurrence relation for $a_n$ as follows. Suppose we have a coloring $C$ of $\{1, 2, \ldots, n\}$ (using only our $k$ colors) and we want to extend it to an allowable coloring $C'$ of $\{1, 2, \ldots, n+1\}$. Then $C$ would certainly have to satisfy condition 1). If $C$ also satisfies condition 2), then we have $k-2$ choices for the color of the new integer $n+1$: it has to be different from the color of $n$, and also different from the color of 1, and as the colors of 1 and $n$ are different, this rules out two of the $k$ colors. So each of the $a_n$ possible allowable colorings $C$ of $\{1, 2, \ldots, n\}$ can be extended to $k-2$ different colorings $C'$, and in this way we get $(k-2)a_n$ allowable colorings $C'$. On the other hand, we can get an allowable coloring $C'$ from a coloring $C$ that satisfies 1) but *not* 2) in $k-1$ different ways, because in choosing the color of $n+1$ there is only one color to avoid. This way we get $(k-1)b_n$ additional allowable colorings $C'$, and so we see that

$$a_{n+1} = (k-2)a_n + (k-1)b_n\,.$$

On the other hand, the colorings counted by $b_n$ are the ones that satisfy 1) and for which the "end" integers 1 and $n$ are the same color. Such a coloring is completely determined by the colors of $\{1, 2, \ldots, n-1\}$; as a coloring of this subset, it satisfies both conditions 1) and 2): because $n-1$ and $n$ cannot have the same color, neither can $n-1$ and 1. Thus we have $b_n = a_{n-1}$, and we now have the second-order recurrence relation

$$a_{n+1} = (k-2)a_n + (k-1)a_{n-1}\,.$$

The solutions of this recurrence relation have the form
$a_n = C_1\lambda_1^n + C_2\lambda_2^n$, assuming that $\lambda_1, \lambda_2$ are distinct real solutions of the characteristic equation

$$\lambda^2 = (k-2)\lambda + (k-1)\,.$$

But this equation is easily seen to have roots $\lambda_1 = -1$ and $\lambda_2 = k-1$, so we have $a_n = C_1(-1)^n + C_2(k-1)^n$. Finally, we need initial conditions to get the values of $C_1, C_2$. For $n = 2$ we can choose any of the $k$ colors for the integer 1 (and all other odd integers) and then any of the remaining $k-1$ colors for 2 (and all other even integers), so we have $a_2 = k(k-1)$. Similarly, we have $a_3 = k(k-1)(k-2)$. Adding the resulting equations

$$C_1 + C_2(k-1)^2 = k(k-1), -C_1 + C_2(k-1)^3 = k(k-1)(k-2)$$

soon yields $C_2 = 1$, $C_1 = k-1$, and the answer follows.

**6**. Evaluate the improper integral

$$\int_0^\infty \frac{25\sin(2017x) - 2017\sin(25x)}{x^2}\,dx\,.$$

(This problem was inspired by an exercise in Spivak's *Calculus*.)

**Answer**. $25 \cdot 2017 \ln\left(\frac{25}{2017}\right) \ = -50425\ln\left(\frac{2017}{25}\right)$.

**Solution**. The numbers $a = 25$, $b = 2017$ are topical for this year's Konhauser, but their actual values don't really affect the calculation, so we'll use $a$ and $b$ for short. It may be tempting to split up the integral as

$$\int_0^\infty \frac{a\sin(bx)}{x^2}\,dx - \int_0^\infty \frac{b\sin(ax)}{x^2}\,dx\,,$$

and then carry out the substitutions $u = bx$, $u = ax$ respectively for the separate parts. Each part becomes

$$\int_0^\infty \frac{ab\sin(u)}{u^2}\,du\,,$$

so the parts would cancel and we would get the answer 0 - if only the parts were convergent! However, as $u$ approaches 0, $\sin(u)/u^2$ behaves like $1/u$, and the integral of this function diverges. So we have to be more careful (even though we'll use the same ideas) and first write the integral (which is improper at both ends) as the sum of two limits:

$$\int_0^\infty \frac{a\sin(bx) - b\sin(ax)}{x^2}\,dx = \lim_{T\to\infty}\int_1^T \frac{a\sin(bx) - b\sin(ax)}{x^2}\,dx + \lim_{t\to 0^+}\int_t^1 \frac{a\sin(bx) - b\sin(ax)}{x^2}\,dx =$$

$$= \lim_{T\to\infty}\left(\int_1^T \frac{a\sin(bx)}{x^2}\,dx - \int_1^T \frac{b\sin(ax)}{x^2}\,dx\right)$$
$$+ \lim_{t\to 0^+}\left(\int_t^1 \frac{a\sin(bx)}{x^2}\,dx - \int_t^1 \frac{b\sin(ax)}{x^2}\,dx\right)$$
$$= \lim_{T\to\infty}\left(\int_b^{bT} \frac{ab\sin(u)}{u^2}\,du - \int_a^{aT} \frac{ab\sin(u)}{u^2}\,du\right)$$
$$+ \lim_{t\to 0^+}\left(\int_{bt}^{b} \frac{ab\sin(u)}{u^2}\,du - \int_{at}^{a} \frac{ab\sin(u)}{u^2}\,du\right)$$
$$= \lim_{T\to\infty}\left(\int_{aT}^{bT} \frac{ab\sin(u)}{u^2}\,du\right) - \int_a^{b} \frac{ab\sin(u)}{u^2}\,du$$
$$+ \int_a^{b} \frac{ab\sin(u)}{u^2}\,du - \lim_{t\to 0^+}\left(\int_{at}^{bt} \frac{ab\sin(u)}{u^2}\,du\right)$$
$$= \lim_{T\to\infty}\left(\int_{aT}^{bT} \frac{ab\sin(u)}{u^2}\,du\right) - \lim_{t\to 0^+}\left(\int_{at}^{bt} \frac{ab\sin(u)}{u^2}\,du\right).$$

We'll show that the first of these limits is zero, but the second is $ab\ln\left(\frac{b}{a}\right)$. For the first limit, note that because $|\sin(u)| \le 1$ for all $u$, we have

$$\left|\int_{aT}^{bT} \frac{ab\sin(u)}{u^2}\,du\right| \le \int_{aT}^{bT} \left|\frac{ab\sin(u)}{u^2}\right|\,du \le \int_{aT}^{bT} \frac{ab}{u^2}\,du = \frac{b-a}{T},$$

which approaches 0 as $T\to\infty$. (Note that if we estimated the second integral in the same way, the bound would go to infinity as $t\to 0^+$.)

For the second limit, note that for $u$ positive and close to zero, we have

$$u - \frac{u^3}{6} \le \sin(u) \le u,$$

and so

$$\int_{at}^{bt} \frac{ab(u-u^3/6)}{u^2}\,du \le \int_{at}^{bt} \frac{ab\sin(u)}{u^2}\,du \le \int_{at}^{bt} \frac{abu}{u^2}\,du.$$

The integral on the right-hand side evaluates to $ab(\ln(bt) - \ln(at)) = ab\ln\left(\frac{b}{a}\right)$, which is independent of $t$. On the other hand, the difference between the "outer" integrals is

$$\int_{at}^{bt} \frac{ab(u^3/6)}{u^2}\,du = \int_{at}^{bt} \frac{abu}{6}\,du,$$

which is easily computed and seen to approach 0 as $t \to 0^+$. Therefore, by the "Squeeze Theorem" we have

$$\lim_{t\to 0^+}\left(\int_{at}^{bt}\frac{ab\sin(u)}{u^2}\,du\right)=ab\ln\left(\frac{b}{a}\right),$$

and the final answer (the difference of the limits) is

$$0-ab\ln\left(\frac{b}{a}\right)=ab\ln\left(\frac{a}{b}\right)=25\cdot 2017\ln\left(\frac{25}{2017}\right),$$

as claimed.

**7**. An eccentric inventor has constructed a row of $N$ light bulbs. Each light bulb has a button switch, but pressing the button does several things: It changes the state of that bulb (from on to off or vice versa) and it also changes the state(s) of that bulb's immediate neighbor(s). (Most bulbs have two neighbors, but the ones at the end of the row have only one.)

a) Are there any values of $N > 1$ for which all the lights can be turned off regardless of the initial configuration (which lights are on and which are off)? If so, what are those values of $N$?
b) For any values of $N$ not covered by part a), find the percentage of possible initial configurations for which all the lights can be turned off, and describe which configurations those are.

**Answer**.
a) Yes, all values of $N$ that are 0 or 1 modulo 3 (that is, of the form $3k$ or $3k+1$).
b) If $N$ is 2 modulo 3, the lights can all be turned off for exactly 50% of the initial configurations. See the solution below for their description.

**Solution**. We first show that regardless of the opening configuration, we can either turn all the lights off or get to the situation where only the rightmost bulb is on. To do this, if any other bulb(s) are on, find the leftmost bulb (say $x$) that is on, and then press the switch for the bulb immediately *to the right* of bulb $x$. As a result, bulb $x$ will now be off, and all bulbs to the left of $x$ will still be off, so the new leftmost bulb that is on (if any) will be further to the right than $x$ was. Repeating this process, eventually we will reach the desired situation in which all bulbs except the rightmost are definitely off.

Now suppose only the rightmost bulb is on and $N = 3k$. Then divide the bulbs that are off into $k$ groups, with one group of 2 bulbs at the extreme left and $k-1$ groups of 3 bulbs. By pressing the switches for the left bulb of the group of 2 and the middle bulb of each group of 3, we can get *all* the bulbs turned on. Once we've achieved that, we can divide all the bulbs into new groups of 3 (starting at one end) and then turn them all off by pressing the switches for the middle bulbs of all the new groups.

Similarly, if only the rightmost bulb is on and $N = 3k+1$, first divide all the other bulbs into $k$ groups of 3 and turn them all on by pressing the switch for the middle bulb of each group. Once all the bulbs are on, regroup them into 2 groups of 2 (one at each end of the row) and $k-1$ groups of 3 to turn them all off.

Finally, suppose that $N = 3k+2$. We can represent a configuration of lights by a row of "bits", say $a_1a_2...a_{3k+2}$, where $a_i = 0$ or 1 according to whether the $i$-th light is off or on. Then the parity of the sum

$$S = a_1 + a_2 + a_4 + a_5 + a_7 + a_8 + ... + a_{3k+1} + a_{3k+2}$$

does not change when any of the switches is pressed. (For example, if the switch for the first bulb is pressed, both $a_1$ and $a_2$ change parity, while the other bits in the sum stay the same; if the switch for the third bulb is pressed, both $a_2$ and $a_4$ change and the others stay the same.) In particular, if we use the process described in the opening paragraph, if $S$ is even we'll end up with all the bulbs turned off, whereas if $S$ is odd we'll end up with $00\ldots01$ and it won't be possible to turn all the bulbs off. $S$ is even for exactly half the possible configurations (given the first $3k+1$ bits, there is one choice for the last bit that makes $S$ even, while the other choice makes $S$ odd), so the lights can all be turned off for 50% of the initial configurations, specifically those for which $a_1 + a_2 + a_4 + a_5 + a_7 + a_8 + \ldots + a_{3k+1} + a_{3k+2}$ is even.

**8**. Find a $3 \times 3$ matrix $A$ with entries from the integers modulo 2, such that $A^7 = I$ but $A \neq I$. (As usual, $I$ denotes the identity matrix, this time with entries modulo 2.)

**Answer**. One such matrix is

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

**Solution 1** (using a rational canonical form). Let $A$ be such a matrix, and let $P(X)$ be the minimal polynomial of $A$ (with coefficients in the field $\mathbb{F}_2 = \{0, 1\}$ of the integers modulo 2). Because $A^7 = I$, we know that $P(X)$ divides $X^7 - 1$. We can factor $X^7 - 1$ (over $\mathbb{F}_2$):

$$X^7 - 1 = X^7 + 1 = (X + 1)(X^3 + X^2 + 1)(X^3 + X + 1).$$

The factors $X^3 + X^2 + 1$ and $X^3 + X + 1$ are both irreducible, because they have degree 3 and it can be checked immediately that they have no roots in $\mathbb{F}_2$. Because $A$ is a $3 \times 3$ matrix, $P(X)$ has degree at most 3, and because $A \neq I$, $P(X)$ cannot be $X + 1$. Thus the only possibilities for $P(X)$ are

$$P(X) = X^3 + X^2 + 1 \quad \text{and} \quad P(X) = X^3 + X + 1.$$

To find a matrix that actually has one of these minimal polynomials, we can use the rational canonical forms corresponding to them, which are

$$A_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{respectively.}$$

As a bonus, we get that every matrix satisfying the conditions of the problem must be conjugate to either $A_1$ or $A_2$.

**Solution 2** (using an extension field). Note that the integers modulo 2 form a field $\mathbb{F} = \mathbb{F}_2$, and that we can get a field extension of degree 3 of this field by adjoining to $\mathbb{F}$ a root of any irreducible polynomial of degree 3 in $\mathbb{F}[X]$. The resulting extension field will have $2^3 = 8$ elements, so its multiplicative group will have 7 elements and so each non-identity element will have order 7. Now the extension field is a 3-dimensional vector space over $\mathbb{F}$, and multiplication by any non-identity element defines a linear transformation from this vector space to itself, whose matrix (with respect to any chosen basis) will have the desired properties.

To carry out this program, we only need to identify an irreducible polynomial of degree 3 over $\mathbb{F}_2$. One such polynomial is $X^3 + X + 1$; it is irreducible because it has no roots in $\mathbb{F}_2$ (check: $0^3+0+1 = 1^3+1+1 = 1 \neq 0$ in $\mathbb{F}_2$) and hence no linear factors, whereas a cubic polynomial can only be reducible if it has a linear factor. Let $\alpha$ be a root of $X^3+X+1$. Then $1, \alpha, \alpha^2$ form a basis of the extension field obtained by adjoining $\alpha$ to $\mathbb{F}$. With respect to this basis, multiplication by $\alpha$ has the matrix given in the answer above: The first column indicates that $\alpha \cdot 1 = \alpha$, the second column indicates that $\alpha \cdot \alpha = \alpha^2$, and the final column indicates that $\alpha \cdot \alpha^2 = \alpha^3 = 1+\alpha$, which follows from the fact that $\alpha$ is a root of $X^3+X+1$. (Remember that in $\mathbb{F}_2$, $-1 = 1$.)

**9**. For values of $x$ sufficiently close to 0, the functions cot and csc can be expressed as

$$\cot x = \sum_{n=0}^{\infty} a_n x^{2n-1}\,, \quad \csc x = \sum_{n=0}^{\infty} b_n x^{2n-1}\,.$$

$$\text{(The coefficients start off }\ a_0 = 1, a_1 = -\frac{1}{3}, a_2 = -\frac{1}{45}, a_3 = -\frac{2}{945}, \ldots$$
$$\text{and }\ b_0 = 1, b_1 = \frac{1}{6}, b_2 = \frac{7}{360}, b_3 = \frac{31}{15120}, \ldots \text{.)}$$

Evaluate the limit

$$\lim_{n\to\infty} \frac{\ln(1+\frac{b_n}{a_n})}{n}\,.$$

**Answer**. $-2\ln 2$ .

**Solution**. If we look at the given values, we see that

$$\frac{b_0}{a_0} = 1\,, \frac{b_1}{a_1} = -\frac{1}{2}\,, \frac{b_2}{a_2} = -\frac{7}{8}\,, \frac{b_3}{a_3} = -\frac{31}{32} \quad \text{and so}$$

$$\ln(1+\frac{b_0}{a_0}) = \ln 2\,, \ln(1+\frac{b_1}{a_1}) = \ln(1/2) = -\ln 2\,,$$
$$\ln(1+\frac{b_2}{a_2}) = \ln(\frac{1}{8}) = -3\ln 2\,, \ln(1+\frac{b_3}{a_3}) = \ln(\frac{1}{32}) = -5\ln 2\,.$$

The pattern seems to be that

$$\frac{b_n}{a_n} = -\frac{2^{2n-1}-1}{2^{2n-1}} \quad \text{and therefore} \quad \ln(1+\frac{b_n}{a_n}) = \ln(1/2^{2n-1}) = -(2n-1)\ln 2\,.$$

If this is true for all $n$, then the desired limit can be rewritten as

$$\lim_{n\to\infty} \frac{\ln(1+\frac{b_n}{a_n})}{n} = \lim_{n\to\infty} \frac{-(2n-1)\ln 2}{n} = \lim_{n\to\infty} \left(-2\ln 2 + \frac{\ln 2}{n}\right),$$

which gives the answer $-2\ln 2$ . So we see that it will be enough to show that $\frac{b_n}{a_n} = -\frac{2^{2n-1}-1}{2^{2n-1}}$ for all $n$; we will do so.

As motivation, suppose we already knew that this is true. Then we would have

$$\begin{aligned}\csc x = \sum_{n=0}^{\infty} b_n x^{2n-1} &= \sum_{n=0}^{\infty} \frac{b_n}{a_n} \cdot a_n x^{2n-1} \\ &= \sum_{n=0}^{\infty} (\frac{1}{2^{2n-1}} - 1) a_n x^{2n-1} \\ &= \sum_{n=0}^{\infty} \frac{a_n}{2^{2n-1}} x^{2n-1} - \sum_{n=0}^{\infty} a_n x^{2n-1} \\ &= \sum_{n=0}^{\infty} a_n (\frac{x}{2})^{2n-1} - \sum_{n=0}^{\infty} a_n x^{2n-1} \\ &= \cot\big(\frac{x}{2}\big) - \cot x\,.\end{aligned}$$

But now we have arrived at something we can check! To put things in logical order, let's start with $\csc x$ again, but this time just use trigonometry, without assuming anything about the series:

$$\begin{aligned}\csc x = \frac{1}{\sin x} &= \frac{1+\cos x}{\sin x} - \frac{\cos x}{\sin x} \\ &= \frac{1 + 2\cos^2(x/2) - 1}{2\sin(x/2)\cos(x/2)} - \cot x \\ &= \frac{\cos(x/2)}{\sin(x/2)} - \cot x = \cot\big(\frac{x}{2}\big) - \cot x\,.\end{aligned}$$

From this identity it follows that

$$\sum_{n=0}^{\infty} b_n x^{2n-1} = \sum_{n=0}^{\infty} a_n (\frac{x}{2})^{2n-1} - \sum_{n=0}^{\infty} a_n x^{2n-1}\,,$$

and comparing coefficients of $x^{2n-1}$ on both sides we have

$$b_n = \frac{a_n}{2^{2n-1}} - a_n = -a_n \frac{2^{2n-1} - 1}{2^{2n-1}}\,,$$

as desired; we are done.

**Comment.** In fact, it can be shown using complex analysis that for every $n \geq 1$,

$$a_n = -\frac{2}{\pi^{2n}} \sum_{k=1}^{\infty} \frac{1}{k^{2n}}, \qquad b_n = -\frac{2}{\pi^{2n}} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2n}}\,.$$

If this is known, we can add these formulas for $a_n$ and $b_n$; the odd-numbered terms will cancel, and the even-numbered terms will be $\dfrac{2}{2^{2n}}$ times the corresponding terms in $a_n$, so that we get

$$a_n + b_n = \frac{2}{2^{2n}} a_n\,.$$

This is easily seen to be equivalent to the relation between $a_n$ and $b_n$ found in the solution above.

**10**. Do there exist uncountably many subsets of $\mathbb{R}$ such that no two of the subsets have an element in common, each of the subsets is uncountable, and each of the subsets is dense in $\mathbb{R}$? (You may use the following lemma, which you may well have seen in the special cases $A = \mathbb{Q}$ and $A = \mathbb{R}$: For any infinite set $A$, the Cartesian product $A \times A$ is equinumerous to $A$.)

**Answer**. Yes.

**Solution**. We start by defining an equivalence relation on $\mathbb{R}$, as follows: For $x, y \in \mathbb{R}$, $x \sim y$ if and only if $y - x$ is rational. It is easy to check that $\sim$ is indeed an equivalence relation; note that for any $x \in \mathbb{R}$, the equivalence class of $x$ is the set $[x] = \{x + q \,|\, q \in \mathbb{Q}\}$. Each of these equivalence classes is dense in $\mathbb{R}$, because approximating any real number $r$ by elements of $[x]$ is equivalent to approximating $r - x$ by rational numbers, which can be done arbitrarily closely (for example, using the decimal expansion of $r - x$). Also, if two equivalence classes are different, they don't have an element in common. However, the individual equivalence classes are *not* uncountable: Each one is equinumerous to $\mathbb{Q}$, and therefore countable.

Let $A$ be the set of all the (different) equivalence classes. Then $A$ is uncountable, because otherwise $\mathbb{R}$, being the union of all the equivalence classes, would be a countable union of countable sets and therefore countable, whereas we know (by Cantor's diagonalization argument) that $\mathbb{R}$ is uncountable. By the lemma, $A$ is equinumerous to $A \times A$. Also, we can write $A \times A$ as the disjoint union of the uncountably many subsets

$$S_a = \{a\} \times A = \{(a, b) \,|\, b \in A\}$$

which are each equinumerous to $A$; therefore, we can write $A$ itself as the disjoint union of uncountably many subsets $T_a$ that are each equinumerous to $A$. (Take the inverse images of the subsets $S_a$ under any bijective function from $A$ to $A \times A$.) Now, for each of the subsets $T_a$, form the subset $\mathbb{R}_a$ of $\mathbb{R}$ that is the union of all the equivalence classes that are elements of $T_a$. We check that the subsets $\mathbb{R}_a$ satisfy the desired conditions.

First of all, there are uncountably many $\mathbb{R}_a$ because $A$ is uncountable. No two of them have an element in common, because the $T_a$ are pairwise disjoint and because different equivalence classes are disjoint. Each of the $\mathbb{R}_a$ is uncountable because each $T_a$ is equinumerous to $A$ (and hence uncountable). And each of the $\mathbb{R}_a$ is dense in $\mathbb{R}$, because every single equivalence class is already dense in $\mathbb{R}$. (As a bonus, the union of all the $\mathbb{R}_a$ is the entire set $\mathbb{R}$.)

**Comment**. It can be shown that the lemma used here is equivalent to the Axiom of Choice.