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Peakless Motzkin paths having a total of $n$ up and level steps

Aaron Laursen* and Ben Hillmann†

1 Introduction

Motzkin paths are a collection of paths consisting of non-negative integer vertices from $(0,0)$ to $(m,0)$ for some given $0 \leq m \in \mathbb{N}$, where each step is one of $\{(1, -1), (1, 0), (1, 1)\}$, corresponding to $\downarrow\nwarrow$, $\rightarrow$, and $\nearrow$, respectively. The “peakless” form requires that no subsequence of the steps in the path is $[(1, 1), (1, -1)]$ (corresponding to a peak $\nearrow\nwarrow$). Let $\mathcal{M}_n$ be the set of all peakless Motzkin paths whose total number of steps of the form $(1, 1)$ and $(1, 0)$ is equal to $n$, and let $M_n = |\mathcal{M}_n|$. We show that $M_n = C_n$, the $n$th Catalan number by showing that our sequence also satisfies the Catalan recurrence $C_0 = 1$ and

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1} \quad \text{for } n \geq 1.$$

2 Examples

The sets $\mathcal{M}_n$ for $n \in \{0, 1, 2, 3, 4\}$ are given in Table 1 below.

---

*Aaron is a Physics and Computer Science major with Mathematics minor at Macalester College. He was born in Minnesota and raised in Illinois, plays waterpolo and the bagpipes, and enjoys a good puzzle.

†Ben is from Minnesota and is a Computer Science and Mathematics major. He enjoys playing games and analyzing the intricacies of winning patterns.
Proof

We prove that $M_n = C_n$ by showing that the sequence $M_n$ satisfies the Catalan recurrence. This entails proving that:

\[ M_0 = 1 \]
\[ M_n = \sum_{k=0}^{n-1} M_k M_{n-k-1} \]
3.1 Initial Conditions

Table 1 above shows that there is exactly 1 peakless Motzkin path with a total of 0 up and level steps. The only possible series of steps is the empty series, since any downward step requires an upward step before it in the path to avoid violating the non-negative vertex prescription in the definition of Motzkin paths. We therefore conclude that \( M_0 = 1 \) and Equation 1 is satisfied.

3.2 Observations

Before proceeding with the proof, we establish a few helpful properties of Motzkin paths. Let \( P \) be a peakless Motzkin path with \( n \) up and level steps.

1. Any subpath of \( P \) that begins at \((i, 0)\) and ends at \((j, 0)\) for \( j \geq i \) is itself a peakless Motzkin path. Such a subpath must not contain peaks, cross the x-axis, or take forbidden steps, since this would necessarily invalidate the original path.

2. If \( Q \subset P \) is a subpath from \((1, 1)\) to \((i, 1)\) which does not intersect \((k, 0)\) for some \( 1 < k < i \), then \( Q \) contains the same sequence of steps as a valid peakless Motzkin path. This subpath is simply a Motzkin path from \((0, 0)\) to \((i - 1, 0)\) shifted up by 1 unit and horizontally by 1 units. The number of up and level steps in \( Q \) must be at least 1 since a length of 0 would create a peak, and can be at most \( n - 1 \) since at least one up step must be used as the first step.

3. If we split a peakless Motzkin path from \( M_n \) into two, disjoint subpaths \( Q_1 \) and \( Q_2 \), then, if we let \( k \) denote the number of up and level steps in \( Q_1 \), \( Q_2 \) must contain \( n - k \).

3.3 Recurrence Relation

In order to prove equation (2), we first split \( M_n \) into two classes. Let \( M_{n,0} \) be those peakless motzkin paths beginning with a step of \((1, 0)\), and let \( M_{n,+1} \) be those beginning with a step of \((1, 1)\). Paths beginning with \((1, -1)\) need not be considered since they violate the non-negative restriction, and therefore \( M_n = M_{n,0} \cup M_{n,+1} \).
3.3.1 Counting $M_{n,0}$

After a step of $(1, 0)$ beginning at $(0, 0)$, a peakless Motzkin path has reached $(1, 0)$, and exactly one up or level step has been taken. Since the remaining steps begin at a height of 0, we can construct $M_{n,0}$ as the series of steps $(1, 0), M_{n-1}$. Therefore,

$$M_{n,0} = 1 \times M_{n-1} \quad (3)$$

3.3.2 Counting $M_{n,1}$

We now consider the second case, $M_{n, +1}$. We note that $M_{n, +1}$ only exists for $n > 1$, since the only valid Motzkin path beginning with a step of $(1, 1)$ and containing no additional steps except for $(1, -1)$ is the path $\bigtriangleup$, which is incompatible with the "peakless" requirement.

Let $n \geq 2$. Consider a path $P \in M_{n, +1}$ with length $m$. Let $(i, 0)$ be the first point after $(0, 0)$ at which $P$ intersects the x-axis. By the second observation, the subpath $Q_1$ from $(1, 1)$ to $(i - 1, 1)$ maps to a valid Motzkin path. By the first observation, the subpath $Q_2$ from $(i, 0)$ to $(m, 0)$ is also a valid peakless Motzkin path.

Let $k > 0$ be the number of up or level steps in the subpath from $(1, 1)$ to $(i - 1, 1)$. Since there is an additional up step before the subpath, the number of up and level steps from $(0, 0)$ to $(i, 0)$ is $k + 1$. This means that the peakless Motzkin path from $(i, 0)$ to $(m, 0)$ contains $n - (k + 1)$ such steps. Therefore, every path $P \in M_{n, +1}$ can be constructed as the series of steps $(1, +1)Q_1(1, -1)Q_2$ for $k \in \{1 \ldots n - 1\}$, $Q_1 \in M_k$, and $Q_2 \in M_{n-(k+1)}$. Which gives:

$$M_{n, +1} = \sum_{k=1}^{n-1} 1 \times M_k \times 1 \times M_{n-(k+1)} \quad (4)$$

3.3.3 Counting all $M_n$

Combining equations (3) and (4) yields

$$M_n = M_{n,0} + M_{n, +1} \quad (5)$$

$$= M_{n-1} + \sum_{k=1}^{n-1} M_k \times M_{n-(k+1)} \quad (6)$$
Using equation (1), this can be rewritten as

$$M_n = M_0 \times M_{n-1} + \sum_{k=1}^{n-1} M_k \times M_{n-(k+1)}$$  \hspace{1cm} (7)$$

Pulling the first term into the summation then yields the desired Equation (2):

$$M_n = \sum_{k=0}^{n-1} M_k \times M_{n-(k+1)}$$  \hspace{1cm} (8)$$

This presents the final piece of the proof that $M_n = C_n$. 
Multisets on $\mathbb{Z}_{n+1}$ whose elements sum to 0

Kyla Martin* and Adam Sirvinskas†

1 Introduction

A set $S$ is defined as a collection of distinct elements, $\{1, 2, 3, \ldots, s\}$, where order does not matter. A multiset is a set where repetition of elements is allowed. The elements of multisets are arranged in non-decreasing order.

Choosing an $r$-element multiset from a set $[s]$ can be thought of as placing balls into boxes. Suppose that we have $r$ identical balls and $s$ distinct boxes. The boxes may be empty and repetition is allowed. We can represent a multiset by the number of balls we put in each box. The number of balls in a box is equal to the number of times you choose that ball. The number of multisets is equal to

$$\binom{r+s-1}{s-1}.$$

We will define $S_n$ to be the collection of all $n$-element multisets of $\mathbb{Z}_{n+1}$ whose elements sum to 0 mod $(n + 1)$. Let $S_n = |S_n|$. We will show that $S_n = C_n$, the $n$th Catalan number, via a direct counting argument.

2 Enumeration for $0 \leq n \leq 4$

Table 1 shows all $n$-element multisets of $\mathbb{Z}_{n+1}$ for $0 \leq n \leq 4$.

---

*Kyla is from Madison, CT and loves the beach.
†Adam is from the Chicago area and likes turtles.
<table>
<thead>
<tr>
<th>$n$</th>
<th>$S_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>00 12</td>
</tr>
<tr>
<td>3</td>
<td>000 013 022 112 233</td>
</tr>
<tr>
<td>4</td>
<td>0000 0014 0023 0113 0122 1112 0244 1144 1333 2233 3444 2224 0334 1234</td>
</tr>
</tbody>
</table>

Table 1: Enumeration of $S_n$ for $0 \leq n \leq 4$

3 Proof

We will now prove that $S_n$ is equal to $C_n$.

We first show that the total number of $n$-element multisets is equal to $\binom{2n}{n}$. Returning to our balls-in-boxes problem, we can clearly see the correlation. The balls in the equation are the number of elements in the multiset, thus $r = n$. The boxes are labeled by the elements of $\mathbb{Z}_{n+1}$. This means that you have $(n + 1)$ positions to put the balls in, thus $s = (n + 1)$. Knowing that the multiset formula is $\binom{r+s-1}{s-1}$, we now substitute for $r$ and $s$, yielding

$$\binom{(n) + (n + 1) - 1}{(n + 1) - 1} = \binom{2n}{n}.$$

We now show how multisets belong to equivalence classes. We define an $n$-element multiset $A$ to be equivalent to an $n$-element multiset $B$, when $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{a_1 + k, a_2 + k, \ldots, a_n + k\}$ for some $k \in \mathbb{Z}_{n+1}$. $A$ is equivalent to $B$ because $\{a_1, a_2, \ldots, a_n\} = \{a_1 + 0, a_2 + 0, \ldots, a_n + 0\}$. If $A$ is equivalent to $B$ then $B = \{a_1 + k, \ldots, a_n + k\}$. So you are adding $k$ to $A$ to get to $B$. So you can add $-k$ to $B$ in order to get $A$. This means that $B$ is equivalent to $A$. If $A$ is equivalent to $B$ and $B$ is equivalent to $C$, then $A$ is equivalent to $C$. In the same way we can add $k$ to $A$ to get to $B$, we can add $j$ to $B$ to get to $C$. That is $C = \{b_1 + j, \ldots, b_n + j\}$. We can go from $C$ to $A$, by adding $-(k + j)$ to $C$. This
shows that \( C \) is equivalent to \( A \). Our relation is indeed an equivalence relation because it is reflexive, symmetric, and transitive.

We now find the size of an equivalence class. For a set \( X = \{x_1, ..., x_n\} \) we define \( \text{sum}(X) = (\sum_{i=1}^{n} a_i) \mod (n + 1) \). Let \( M = \{a_1, a_2, ..., a_n\} \) be our original set with \( \text{sum}(M) = 0 \). Let \( M_k = \{a_1 + k, a_2 + k, ..., a_n + k\} \). We have

\[
\sum(M_k) \equiv \left( \sum_{i=1}^{n} a_i + k \right) \mod (n + 1) = \left( \sum_{i=1}^{n} a_i + nk \right) \mod (n + 1)
\]

\[
= nk \mod (n + 1) = ((n + 1)k - k) \mod (n + 1) = -k,
\]

so these values are all distinct. Because we are dealing with \( S_n \), there will only be \( n + 1 \) possibilities for the value of \( k \). This means the size of an equivalence class is \( n + 1 \).

We will now show that one multiset per equivalence class belongs to \( S_n \). The \( \text{sum}(M_k) \) is 0 if and only if \( k = 0 \), because \( -(0) \mod (n + 1) = 0 \). Thus, there is only one multiset per equivalence class that belongs to \( S_n \).

We can now find \( S_n \). Dividing the total number of multisets of \( \mathbb{Z}_{n+1} \), \( \binom{2n}{n} \), by the size of the equivalence class, \( (n + 1) \), yields \( \binom{2n}{n} \frac{1}{n+1} \). By definition, this is equal to the \( n \)th Catalan number \( C_n \). \( \square \)
Dyck Paths of Length $4n$ with Descents of Length 2

Devin Bjelland,* Arianna Hesterberg†

1 Introduction

A Dyck path is a sequence of up and down slopes of length one with the property that the path starts and ends on the $x$-axis and never goes below the $x$-axis. Let $A_n$ denote the set of Dyck paths of length $4n$ with descents of length 2 and let $A_n = |A_n|$ We will show that $A_n = C_n$, the nth Catalan number.

2 Examples

The sets $A_n$ for $1 \leq n \leq 4$ are as follows:

---

*Devin is a freshmen from Southern Minnesota who enjoys programming computers and pictures of cats.  
†Arianna, a sophomore from Saint Paul, MN, loves to play volleyball and solve Sudoku puzzles.
In order to show that $A_n$ is counted by the Catalan numbers, we construct a bijection to the set $B_n$ of mountain ranges of size $2n$, which are known to be counted by the Catalan numbers. Let $B_n = |B_n|$.

### Table

<table>
<thead>
<tr>
<th>$n$</th>
<th>$A_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td><img src="image1.png" alt="Mountain Range 1" /></td>
</tr>
<tr>
<td>2</td>
<td><img src="image2.png" alt="Mountain Range 2" /></td>
</tr>
<tr>
<td>3</td>
<td><img src="image3.png" alt="Mountain Range 3" /></td>
</tr>
<tr>
<td>4</td>
<td><img src="image4.png" alt="Mountain Range 4" /></td>
</tr>
</tbody>
</table>

### 3 Bijection

In order to show that $A_n$ is counted by the Catalan numbers, we construct a bijection to the set $B_n$ of mountain ranges of size $2n$, which are known to be counted by the Catalan numbers. Let $B_n = |B_n|$.
3.1 Definition of $f$

We define $f : A_n \rightarrow B_n$ as follows. The mapping preserves all upward segments that are not immediately followed by a descent. The mapping transforms upward segments with the following descent of length two into a single descent of length 1. For example:

<table>
<thead>
<tr>
<th>$x \in A_n$</th>
<th>$f(x) \in B_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Diagram" /></td>
<td><img src="image2.png" alt="Diagram" /></td>
</tr>
<tr>
<td><img src="image3.png" alt="Diagram" /></td>
<td><img src="image4.png" alt="Diagram" /></td>
</tr>
<tr>
<td><img src="image5.png" alt="Diagram" /></td>
<td><img src="image6.png" alt="Diagram" /></td>
</tr>
</tbody>
</table>

There are an equal number of descents and ascents in a Dyck path. The mapping $f$ takes an ascent followed by a descent of length 2 to a single descent. Therefore, a Dyck path of length $4n$ is transformed into a path of length $2n$. The resultant path still never goes below the $x$-axis because the height change of an ascent followed by a descent length 2 descent is equivalent to the height change of a single descent. This shows that $f$ is well defined because it maps an element of $A$ to and element of $B$.

3.2 Definition of inverse $g$

Let $g : B_n \rightarrow A_n$. We define $g$ to preserve ascents and transform descents into an ascent followed by a descent of length 2. Since every element is either an ascent or descent, $g$ is well defined. The resulting path has length $4n$ since each of the $n$ descents is transformed into a subpath of length 3. The resulting path is a Dyck path since an ascent followed by a descent of length 2 has the same effect as a
single descent on the elevation. This proves that \( g \) is well defined.

### 3.3 Proof that \( f \) and \( g \) are inverses

#### 3.3.1 Proof that \( f \circ g = I_B \)

The mapping \( g \) preserves ascents and \( f \) preserves ascents that are not followed by a descent. Since \( g \) adds an ascent before each descent, ascents that are in the original path are preserved. Therefore \( f \circ g \) preserves the ascents. And since \( g \) transforms a descent into an ascent followed by a descent of length 2 and \( f \) does the reverse on those subpaths, we conclude \( f \circ g = I_B \).

#### 3.3.2 Proof that \( g \circ f = I_A \)

The same argument applies to show that \( g \circ f \) preserves the ascents not followed by a descent. And for the remaining up slopes, \( f \) takes an ascent followed by a descent of length 2 to a single descent while \( g \) does the reverse. Therefore \( g \circ f = I_A \).

### 3.4 Conclusion

Since \( f \) is an invertible function, we know that \( f \) is a bijection. Therefore \( A_n = B_n = C_n \) and we conclude the Dyck paths of length \( 4n \) with descents of length 2 are counted by the Catalan numbers.
Noncrossing Increasing Trees

Andy Walsh*

1 An Introduction

Let $A_n$ be the set of noncrossing, increasing trees on the vertex set $[n + 1]$. We represent a tree in $A_n$ as follows. Its vertices are arranged in increasing order around a circle such that no edges cross in their interior, and such that all paths from vertex 1 are increasing. We will prove that the number $A_n = |A_n|$ is equal to $C_n$, the $n$th Catalan number.

2 Examples

Figure 1 demonstrates $A_n$ for $0 \leq n \leq 4$.

*Andy was born and raised in Minneapolis. He enjoys gaming, socializing and arguing.
<table>
<thead>
<tr>
<th>$n$</th>
<th>Noncrossing increasing trees of length $n + 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td><img src="https://example.com/diagram0.png" alt="Diagram" /></td>
</tr>
<tr>
<td>1</td>
<td><img src="https://example.com/diagram1.png" alt="Diagram" /></td>
</tr>
<tr>
<td>2</td>
<td><img src="https://example.com/diagram2.png" alt="Diagram" /></td>
</tr>
<tr>
<td>3</td>
<td><img src="https://example.com/diagram3.png" alt="Diagram" /></td>
</tr>
<tr>
<td>4</td>
<td><img src="https://example.com/diagram4.png" alt="Diagram" /></td>
</tr>
</tbody>
</table>

Figure 1: An enumeration of noncrossing, increasing trees.
3 Proving the Catalan Recurrence

The Catalan numbers satisfy the following recurrence:

\[
\begin{align*}
C_0 &= 1, \\
C_{n+1} &= \sum_{i=0}^{n} C_i C_{n-i} \text{ for } n \geq 1.
\end{align*}
\]

We prove that the numbers \( A_n \) also satisfy this recurrence relation.

For \( n \geq 1 \), there will always be a line connecting node 2 to node 1 because the only number less than it is 1 in every diagram. Therefore, for the sake of our recurrence relation, we can ignore the (1) – (2) connection.

With this connection gone, we have a forest with two tree components. Let \( B_n \) be the family of the 2-forests that result from removing the (1)–(2) connection from the trees in \( A_n \). Every 2-forest in \( B_n \) will still be distinct.

3.1 The case \( n = 4 \)

For clarity, we will start by looking at the 2-forests in \( B_4 \). We see that node 2 is isolated in some of them. Since there no connection left involving node 2, we can ignore it. It becomes apparent that the other tree in \( B_4 \) is similar to \( A_3 \), the only difference being the names of the nodes. If we change the indexing of the nodes (so that node 3 is renamed node 2, node 4 is then node 3, and node 5 is then node 4), we see that these trees are indeed identical to \( A_3 \). The disjoint node 2 would then be equivalent to \( A_0 \), as shown in Figure 2. By the product rule, we see that the number of different arrangements of \( B_4 \) in which node 2 is disjoint is equal to \( A_0 * A_3 \).

![Figure 2: Examples of 2-forests of type \( A_3 * A_0 \).](image-url)
The next part of the recurrence can be linked to all $B_4$ such that node 2 is only connected to node 3. The (2) – (3) connection can be thought of as a tree of the $A_1$ family and the tree involving 1, 4, and 5 can be thought of as a tree from the $A_2$ family with a reindexing of the nodes following the same logic as applied in the previous paragraph, as shown in Figure 3. The product rule shows that the total number of these trees equals $A_1 \ast A_2$.

![Figure 3: Examples of 2-forests of type $A_2 \ast A_1$.](image)

Another type of arrangement found in $B_4$ is a 2-forest such that node 1 is only connected to node 5, and nodes 2, 3, and 4 are connected to each other, see Figure 4. The product rule can again be used to show that there are $A_1 \ast A_2$ arrangements that fit these criteria.

![Figure 4: Examples of 2-forests of type $A_1 \ast A_2$.](image)

The final type of arrangement found in $B_4$ is the arrangement in which node 1 is disjoint, and nodes 2, 3, 4, and 5 are connected, forming a tree equivalent to $A_3$, see Figure 5. Hence, there are $A_0 \ast A_3$ different arrangements of this type.
3.2 The general case

In the general case, if we have \([n + 1]\) nodes, then there are \([n]\) choices for the lowest-number connection for node 1 (after removing the trivial connection to node 2). These choices are \(\{3, 4, 5, ..., n + 1, \emptyset\}\), where \(\emptyset\) signifies that node 1 has no other connections. This connection is what determines the sizes of each tree in our 2-forest \(B_n\), as will be explained.

Let us say that the lowest-numbered node that node 1 connects to in \(B_n\) is node \(k\). Then, the set of nodes \(\{2, 3, 4, ..., k - 1\}\), and only these nodes, must form a tree, while nodes \(\{1, k, k + 1, k + 2, ..., n\}\) form the other tree. This is due to the stipulation that the connections must be non-crossing. If any node contained in \(\{2, 3, 4, ..., k - 1\}\) were to be connected to any node greater than \(k\), then it would cross the \((1)-(k)\) connection, and it would not be a member of the set that we are counting.

So, using the lowest-valued connection from node 1 as the determinator of the size of our trees in the 2-forest, we see that if the size of the tree containing node 2 is \(k\), then the size of the remaining tree is \(n - k\). Using the product rule for each \(B_n\) and summing up all different possibilities for \(k\) gives us the size of \(A_n\).

This is equivalent to

\[
A_0 = 1, \\
A_{n+1} = \sum_{k=0}^{n} A_k A_{n-k} \text{ for } n \geq 0,
\]

where the value \(A_0 = 1\) comes from Figure 1. This shows that the family \(A_n\) satisfies the Catalan recurrence. Therefore, \(A_n = C_n\).
Zero Sum Sequences with Positive Partial Sums

Jeffrey Lyman* and Scott Dixon†

1 Introduction

Consider $P_n$ the set of sequences $p_1, p_2, p_3, \ldots, p_n$ where

1. $p_1 + p_2 + \ldots + p_n = 0$
2. $-1 \leq p_i \leq n - 1$
3. $\sum_{k=1}^{i} p_k \geq i$

We will prove that $P_n = |P_n| = C_n$, the $n$th Catalan number.

2 Enumeration

Here are the values of $P_n$ for $0 \leq n \leq 4$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$P_n$</th>
<th>$C_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\emptyset$</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>[0]</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>[0,0] [1,-1]</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>[0,0,0] [1,-1,0] [1,0,-1] [0,1,-1] [2,-1,-1]</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>[0,0,0,0] [0,0,1,-1] [0,1,0,-1] [1,0,1,0] [1,0,0,-1] [1,0,-1,0] [1,-1,0,0] [1,-1,1,-1] [1,1,-1,1] [0,2,-1,-2] [2,0,-1,-1] [2,-1,0,-1] [2,-1,1,0] [2,-1,1,-1]</td>
<td>14</td>
</tr>
</tbody>
</table>

*Jeffrey is a Sophomore CS major at Macalester College. He enjoys choir and sweaters.
†Scott is a Sophomore Math major from North Carolina. He doesn’t like writing footnotes.
3 Proof

3.1 Introduction and Definitions

We prove that $P_n = C_n$ by constructing a bijection between $P_n$ sequences and a known Catalan family $M_n$, the monotonic paths that do not pass above the main diagonal. The set $M_n$ consists of all paths contained in the $n \times n$ grid that begin in the lower left corner of the grid and finish in the upper right corner while only moving rightwards or upwards. Figure 1 shows the paths in $M_3$.

In order to define a function $f : P_n \rightarrow M_n$, we will consider an intermediate family of structures. Let $T_n$ be the set of transformed sequences $(t_1, t_2, \ldots, t_n)$ that obey the following rules:

1. $t_1 + t_2 + \cdots + t_n = n$
2. $0 \leq t_i \leq n$
3. $\sum_{k=1}^i t_i \geq i$

We define the functions $g : P_n \rightarrow T_n$ and $h : T_n \rightarrow M_n$ such that $f = (h \circ g)$.

3.2 Proof that $g$ is a bijection

We start by considering the function $g : P_n \rightarrow T_n$. We will define $g$ as a function that adds one to each element of $p \in P_n$. The function $g$ is bijective if and only if

1. $\sum_{k=1}^n p_k = 0 \iff \sum_{k=1}^n t_k = n$
2. $-1 \leq p_i \leq n - 1 \iff 0 \leq t_i \leq n$
3. $\sum_{k=1}^i p_k \geq 0 \iff \sum_{k=1}^i t_k \geq i$
The function $g$ meets the first condition because $\sum_{k=1}^{n} t_k = \sum_{k=1}^{n} (p_k + 1) = n + \sum_{k=1}^{n} p_k = n$

The function $g(p)$ satisfies the second rule because there can be at most $(n - 1) - 1$’s in a $p$. For $p = (p_1, \ldots, p_n)$, we have $-1 \leq p_i \leq n - 1$ for $1 \leq i \leq n$. Therefore for $g(p) = a = (a_0, \ldots, a_{n-1})$ where $0 \leq a_i \leq n$ for $1 \leq i \leq n$.

Furthermore, $g$ satisfies the third rule. The $i^{th}$ partial sum of $g(p) = a = \sum_{k=1}^{i} a_k = \sum_{k=1}^{i} (p_n + 1) = i + \sum_{k=1}^{n} p_n$. We know that $\sum_{k=1}^{n} p_n \geq 0$ so the $i^{th}$ partial sum of $g(p) \geq i$. Thus $g$ is surjective because for every $p \in \mathcal{P}_n$, $g(p)$ creates a $t \in \mathcal{T}_n$. Since $g$ meets our three conditions it is a bijection.

### 3.3 Proof that $h$ is a bijection

We define $h : \mathcal{T}_n \rightarrow \mathcal{M}_n$ as follows. The sequence $t = (t_1, \ldots, t_n) \in \mathcal{T}_n$ is mapped to the monotonic path beginning at $(0,0)$ that moves right $t_1$ steps, and then up one step, followed by $t_2$ steps rightward and one up step, and so on. For example, $(3,1,0,0,1) \in \mathcal{T}_5$ is mapped to the monotonic path in Figure 2:

![Figure 2: The sequence $(3,1,0,0,1) \in \mathcal{T}_5$ maps to this monotonic path in $\mathcal{M}_5$.](image)

We claim that $h$ is well defined. For every $t \in \mathcal{T}_n$, $h(t)$ describes a path from $(0,0)$ to $(n,n)$ since the height of the $h(t)$ is the number of elements in $\mathcal{T}_n = n$ and the distance to the right is determined by the sum of the $\mathcal{T}_n$ series which is also be equal to $n$.

The image $h(t)$ paths will always be northeast since $0 \leq t$, and $h$ does not describe any downward or leftward motion.

In order to cross the main diagonal, the total number of steps to the right must be less than the total number of steps taken up. The height of a monotonic $h(t)$
after \( i \) right steps is \( i \) and the distance to the right after \( i \) steps will be the partial sum of the first \( i \) terms. By definition, the partial sums of the first \( i \) elements of \( T_n \) is at least \( i \). Therefore \( h(t) \) will never cross the main diagonal. This shows that \( h(t) \) is a monotonic path and so \( h \) is a well defined.

We claim that \( h \) is surjective. Consider a path \( m \in M_n \), now consider the sequence \( s \) where \( s_i = \) the number of right-steps between the \( i^{th} \) and the \( (i+1)^{st} \) up-step. 

\[ |s| = n \] because there are \( n \) up-steps in the monotonic path. Furthermore \( \sum_{k=1}^{n} s_k = n \) because there are \( n \) right-steps in a monotonic path.

Monotonic paths can’t move backwards or run off the grid so \( 0 \leq s_i \leq n \).

Monotonic paths also can’t travel further up than right so \( \sum_{k=1}^{i} s_i \geq i \). Therefore \( s \in T_n \) and so \( \forall m \in M_n \) there is a \( t \in T_n \) and so \( h \) is surjective.

We claim that \( h \) is injective. Consider two series \( a, b \in T_n \) such that \( a \neq b \). Consider the first index \( i \) such that \( a_i \neq b_i \). This means that the two paths must take a different number of right-steps at height \( i \). Therefore \( h(a) \neq h(b) \). Thus, for all \( a, b \in T_n \), we have \( h(a) = h(b) \rightarrow a = b \). So \( h \) is injective.

Given that \( h \) is both injective and surjective, \( h \) must be bijective by the definition of a bijection.

### 3.4 Conclusion

Our function \( f : P_n \rightarrow M_n \) is a bijection because \( f = h \circ g \) and \( h \) and \( g \) are bijections. Therefore \( |P_n| = |M_n| = C_n \), the \( n \)th Catalan number.
Plane Trees with Special Left Internal Vertices

Xenia Ewing* and Rebecca Gold†

1 Introduction

Let $T_n$ be the set of plane trees with $n$ internal vertices such that each vertex has at most 2 children and each left child of a vertex with two children is an internal vertex. Instead of the traditional definition of an internal vertex as one with degree at least two, here an internal vertex is any vertex with a child, including the root. In other words, the root counts as an internal vertex. This may seem a bit strange, but bear with us! It will make sense in time.

We show that there is a bijection between these trees and the balanced sequences of 1’s and −1’s of length $2n$.

2 Examples

Table 1 shows some examples of these planar trees with special left internal vertices. Our example set is from $n = 0$ to $n = 4$. There are 1, 1, 2, 5, and 14 trees respectively for each row.

---

* Xenia Ewing a psychology major and computer science minor from Hawaii (but is technically from Illinois). She enjoys dabbling in math and likes penguins.

† Rebecca Gold is a Computer Science major from the suburbs of New York City. She is very passionate about the gender gap in Math and Computer Science. She also believes that ducks and Math are intrinsically linked.
\[ n \mid \text{Plane trees with } n \text{ internal vertices (specifications noted in introduction).} \]

<table>
<thead>
<tr>
<th>0</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td><img src="image1" alt="Tree 1" /></td>
</tr>
<tr>
<td>2</td>
<td><img src="image2" alt="Tree 2" /></td>
</tr>
<tr>
<td>3</td>
<td><img src="image3" alt="Tree 3" /></td>
</tr>
<tr>
<td>4</td>
<td><img src="image4" alt="Tree 4" /></td>
</tr>
</tbody>
</table>

Table 1: The sets \( \mathcal{T}_n \) for \( 0 \leq n \leq 4 \)

3 Bijection

We now detail the bijection \( f \) of \( \mathcal{T}_n \) to the known Catalan family \( B_n \) of balanced sequences of 1 and \(-1\) (which we will simply denote as “\(-\)” for clarity) of length \( 2n \).
3.1 Mapping our Trees to Balanced Sequences

We perform a depth-first search through the tree, processing left children before right children. We label each edge as follows:

1. Mark as $1-$ if the vertex below the edge has no siblings.
2. $1$ if the vertex below it is a left child.
3. $-1$ if the vertex is a right child.

This gives you an ordering of $1$’s and $-1$’s. The first entry is positive because either the root has one child, or its left child of the root is processed first. The last entry is always negative because the final processed edge either has no siblings or is a right child. Furthermore, the sequence is balanced because a node will either have one or two children. If it has one child, that child gets two symbols ($1-$). If it has two children, they each get their own symbol (a $1$ or a $-1$). Therefore, we will always have an even number of symbols. These two properties combined satisfy the property of a balanced sequence of $1$’s and $-1$’s of length $2n$. We now have a direct mapping to the Balanced Sequences of $1$’s and $-1$’s.

For example, we might have the following tree where $n = 3$ and each internal vertex has one child. Such a tree has mapping $1-1-1-$, like so:

```
  +---------+
  |        |
  |        |
  +--------+
    |      |
   1-     1-
    |      |
   1-     1-
    |      |
   1-     1-
```

Similarly, if we have a more difficult tree with a root with one child which in turn has two children and the left child also has a child, it maps to $1-11-1-$, like so:

```
  +---------+
  |        |
  |        |
  +--------+
    |      |
   1-     1-
    |      |
   1-     1-
    |      |
   1-     1-
```
3.2 Proving The Mapping Injective

When we have distinct trees, the sequences they correspond to differ at the same points as the trees differ during the DFS.

These two trees start off looking the same, as we highlight with the non-dotted lines in these two trees. Those highlighted parts correspond to the underlined parts of the balanced sequences of these two trees: $11--1$ and $11--11$. This convergence proves that we will get a unique tree for every binary sequence and a unique binary sequence for each tree and with this we prove that $\mathcal{T}_n$ and $\mathcal{B}_n$ are injective.

3.3 Proving The Mapping Surjective

The mapping process works backward to help us see that this argument is surjective. If we are given $b \in \mathcal{B}$ (a sequence in $\mathcal{B}_n$), we can find a tree $t \in \mathcal{T}$ such that $f(t) = b$. We do this through the following backwards mapping process:

1. We count the number of $1$ and $-$ symbols and divide by 2, then we get our value for $n$. Then we follow these steps:
   1. If the sequence begins with a $1-$, then the tree starts with a single line down.
   2. If it starts with a $11$, then the first place we are going to go is the left child. We will eventually address the right child. Continue parsing the sequence of $1$’s and $-$’s starting from the second $1$ of those two $1$s.
   3. Continue parsing through the sequence considering $1-$ to be a straight shoot down or a left and a right child, $11$ to indicate a left child and continue parsing from the second $1$. Keep in mind that if you hit a $---$ then you should go back up the tree to the most recent point where you saw that a left child would be branched and begin with that implied right child.
For example, consider the sequence $11 \overline{1}1$. For emphasis, we underline symbols that we are processing. Our first step is to consider $\overline{11}1$. This starts with a left branch (the right branch is implied to be something we’ll get to later).

So now it’s $\overline{11}1$. This would put a straight line down from that left branch.

Then it’s $\overline{H}1$. Here we go back to the implied right branch.

Finally it’s $11 \overline{1}$. This just means a straight line down from our current branch, which happens to be the right branch.

The tree $t$ is a member of $T_n$ such that $f(t) = b$. This proves that $f$ is surjective. With this bijection (proven by showing that these families are injective and surjective), we have demonstrated that $T_n$ maps to a known Catalan family and therefore, we have proven that it maps to the Catalan numbers.
Non-Intersecting Odd-Even Arcs
Lucas Gagnon\textsuperscript{*} and Jie Shan\textsuperscript{†}

1 Introduction

Let $A_n$ be all $n$-graphs of non-intersecting odd-even arcs. These graphs consist of $2n - 2$ nodes in a line numbered starting with 1. Nodes are connected by nonintersecting arcs such that the index of the left end node (arc beginning) of each arc is odd and the index of the right end node (arc end) is even. Isolated nodes are allowed. Additionally, let $A_n = |A_n|$. We prove that $A_n = C_n$, using a bijective proof.

2 Non-intersecting Odd-Even Arcs for 2n-2 Nodes

Below are examples of non-intersecting odd-even arcs, for $n = \{1, 2, 3, 4\}$, listed with the number of different graphs for the chosen $n$, $C_n$.

$n = 0, C_0 = 0$

\[\emptyset\]

$n = 1, C_1 = 1$

Since $2n - 2 = 0$, no node is displayed, and no connections can be made, there is one way to do nothing.

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\textsuperscript{†}Jie is currently a senior at Macalester College from Shanghai, China. He loves technology and playing soccer.
\[ n = 2, C_2 = 2 \]
\[
\begin{array}{c}
\ldots \\
\end{array}
\]

\[ n = 3, C_3 = 5 \]
\[
\begin{array}{c}
\ldots \\
\ldots \\
\ldots \\
\end{array}
\]

\[ n = 4, C_4 = 14 \]
\[
\begin{array}{c}
\ldots \\
\ldots \\
\ldots \\
\ldots \\
\ldots \\
\ldots \\
\ldots \\
\end{array}
\]

3 Proof

We will use a bijection to prove that \(|A_n|\) is counted by the Catalan numbers. We create a mapping \(M\) from \(A_n\) to the set \(B_n\) of balanced sequences of \(n\) 1’s and \(n\) \(-1\)’s and prove that \(M\) is a bijection. This shows that \(A_n = C_n\) for all \(n\).

3.1 The Mapping

The mapping \(M\) is given by the following algorithm.
1. Label each odd node where an arc begins with a 1.
2. Label each even node where an arc ends with a $-1$.
3. Label each odd unconnected node with a $-1$.
4. Label each even unconnected node with a 1.
5. We now have a sequence of $2n - 2$ 1’s and $-1$’s. At the beginning of the sequence, write a 1. At the end, write a $-1$.

3.2 An Example of the Mapping $M$

Consider the graph below, which is a member of non-intersecting odd-even arcs when $n = 4$.

First we place a 1 below each node where an arc begins:

\[ 1 \quad 1 \quad \ldots \quad \ldots \]

Then a $-1$ below each node where an arc ends:

\[ 1 \quad -1 \quad 1 \quad -1 \quad \ldots \quad \ldots \]

Next, under each unconnected odd node, we write down a $-1$:

\[ 1 \quad -1 \quad 1 \quad -1 \quad -1 \quad \ldots \quad \ldots \]

and a 1 under each unconnected even node:

\[ 1 \quad -1 \quad 1 \quad -1 \quad -1 \quad 1 \]

Leaving out arcs and points, a sequence of $2n - 2$ 1’s and $-1$’s with $n - 1$ of each is obtained:

\[ 1, -1, 1, -1, -1, 1 \]

Finally, adding a 1 at the front and a $-1$ at the end concludes our mapping:

\[ 1, 1, -1, 1, -1, -1, 1, -1 \]

The output of the mapping is a member of balanced sequence of 1’s and $-1$’s when $n = 4$, satisfying the requirement that any partial sum in the sequence is positive. We will see below that the mapping can also be easily reversed.
3.3 $M$ is Well Defined

We will now show that $M$ is a well defined function. If $a \in A_n$, then $b = M(a)$ is a sequence of $2n$ $1$’s and $-1$’s. To confirm that $b \in B_n$, we still must show that

1. the partial sum of this sequence is at no point less than 0, and
2. this sequence sums to zero.

In this proof, we will discuss the labels for the nodes of $a$, given by steps 1 through 4 of our algorithm $M$. We define $b'$ as the sequence of all node labels for $b$. Thus, $b'$ is a sequence of length $2n - 2$, as it includes all numbers of $b$, save the $1$ at the beginning and the $-1$ at the end. Then, the partial sums of $b'$ may be as low as $-1$ without violating (1).

Now, we prove (1). We start by pairing the $2k - 1$ and $2k$ nodes of $a$ for $1 \leq k \leq n - 1$. The first node in this pairing is an odd node, and the second is an even node. Then, each pairing could be labeled $(1,1)$, or $(-1,1)$ or $(1,-1)$, or $(-1,-1)$ by $M$, as shown in the diagram below.

```
Case 1
1 1
-1 1

Case 2
1 -1

Case 3
-1 -1

Case 4
```

Therefore, each pair adds either $+2$, $0$, or $-2$ to the partial sum of $b'$. We need not worry that a pair that adds $0$ actually subtracts $1$ and then adds $1$, because the partial sum of node labels may be $-1$. Thus, $M$ can only violate (1) when it adds $-2$ to the partial sum.

Consider a pair $2k - 1, 2k$ that contributes $-2$ to the partial sum. This happens when node $2k - 1$ is unconnected and node $2k$ is the end of an arc, as shown in case 4 of the above diagram. However, an arc ending at $2k$ in $a$ can only exist when there is a corresponding arc beginning at $2j - 1$ that connects to $2k$, where $j < k$. Then the paired even node of $2j - 1$, node $2j$, must be unconnected. Indeed, node $2j$ can only be connected to earlier odd nodes, but cannot connect with any such node without violating the non-intersection property of $A_n$. Thus, the pair $2j - 1, 2j$ must add $2$ to the partial sum of $b'$, as $2j - 1$ would be labeled $1$, and $2j$ would also be labeled $1$. Such a pairing is shown in case 1.

Thus, we can associate each $-2$ value pair $2k - 1, 2k$ to a preceding $+2$ value pair $2j - 1, 2j$. From this we conclude that the partial sum of $b'$ will always have
2 added to it before $-2$ is added. This means that the partial sum of $b'$ at $2k$ must have at least as many 2 value pairs as $-2$ value pairs for all $1 \geq k \geq n - 1$, and so the partial sum will be at least 0. It follows that the partial sum at $2k - 1$ is not less than $-1$, as the partial sum changes by $\pm 1$ for each entry. Thus, we have shown (1).

We now prove (2). In $b'$, each +2 pair has a matching $-2$ pair. All other pairs add 0 to the sum of $b'$. Therefore the sum of $b'$ must be 0. Then, the sum of $b$ must be $1 + 0 - 1 = 0$ because of the 1 at the beginning of $b$ and the $-1$ at the end of of $b$ that are not in $b'$. Thus, $b$ is balanced, and $M$ is well defined.

3.4 Proof of Injectivity

Let $a, a' \in A_n$ such that $M(a) = b = M(a')$ for some $b \in B_n$. We will show that $a'$ must be equal to $a$.

Consider any entry in $b$, with the exception of the first or last entry. By the algorithm for $M$, we know that this number corresponds with one node of $a$ and $a'$. Further, we know that this node must be in one state, connected or unconnected, which we can determine based on the values of the corresponding entry in $b$. We may generalize this process to all $(n - 2)$ allowable entries of $b$ so that we may discern the ‘connection states’ of each node in $a$ and $a'$.

The arcs in $A$ must be non-intersecting and can only begin at an odd node and end at an even node. This means that no two distinct $n$-graphs have the same connection states. Therefore, as $a$ and $a'$ have the same connection state, they must be the same $n$-graph of non-intersecting odd-even arcs. Thus, $M$ is injective.

3.5 Proof of Surjectivity

Let $b \in B_n$. We will show that there exists an $a \in A_n$ such that $M(a) = b$.

As all elements of $B_n$ begin with 1 and end with $-1$, we define the truncated sequence $b'$ to be $b$ without its first or last entries. The sequence has length $2n - 2$ and $b'$ must still have an equal number of 1’s and $-1$’s. The partial sum property of $B_n$ mandates that the partial sum of $b'$ is never less than $-1$.

Now, we show that each $-1$ can be associated with an earlier 1. We will use the same even odd pairing process as we did in the above section, pairing the
2k − 1 and 2k entries of b', for 1 ≤ k ≤ n − 1. We show that any partial sequence of b' contains at least as many +2 pairs as −2 pairs. Consider the number of 1’s in odd indicies and −1’s in even entries. We divide all −1’s in even entries into two types. First, a −1 in entry 2k could be immediately preceded by a 1 in the 2k−1. Otherwise, it is immediately preceded by a −1. In the first case, we associate our −1 with the 1 immediately before it. In the second case, our −1 is part of a −2 value pair. We have already shown that each −2 value pair in b' can be associated with a +2 value pair somewhere earlier in b'. Therefore, we may pair these two −1’s with the two 1’s of the +2 value pair. This proves that every −1 can be associated with a 1 earlier in b'.

We will now turn b' into an element of A_n. We start by replacing 1’s in odd indices with arc beginnings, −1’s in even indices with arc endings, and all other numbers in b with lone nodes. Next, we find the an arc beginning and an arc ending with no unconnected arc beginnings or endings between them, and connect these two nodes. We repeat this process recursively until we cannot find any such pair. We will refer to this method as M' for the duration of this proof.

This mapping M' results in an element of A_n. By the very definition of M', we know that it results in a graph with 2n − 2 nodes. The graph has an equal number of arc beginnings (1’s on odd indicies) and arc endings (−1’s on even indicies), and the number of arc beginnings can never be less than the number of arc endings. All arc beginnings and endings must be connected as there are an equal number of arc beginnings and endings, and from the leftmost node, the number of arc beginnings is always greater than the number of arc endings. Thus, we have a graph that is a member of A_n.

We point out to that M' is the algorithm of M backwards, so that if M'(b) = a ∈ A_n, then M(a) = b. Thus, we have proved that M is surjective. In the next section we show an example the process of M'.

3.6 Finding an a for b such that M(a) = b

We will start with the sequence

1, 1, −1, 1, −1, −1, 1, −1

from B_4. First, remove the first 1 and the last -1 from the sequence, resulting in the new sequence

1, 1, −1, 1, −1, −1, 1
Then, draw nodes over every entry.

\[
1, -1, 1, -1, -1, 1.
\]

Now, start arcs at odd nodes with entries of 1, and draw arc endings at even nodes with entries of -1.

Finally, we will connect each arc beginning to the first arc ending such that the number of arc beginnings underneath the closed arc is equal to the number of arc endings underneath the arc, starting from the right.

We now have an 4-graph of non-intersecting odd-even arcs. Since we oM' is the reverse process of M, if we apply M again to the above graph, we will get back to the member in \( B_4 \) at the beginning of the section.

**Conclusion**

By constructing a bijective mapping from \( A \) to a known member of the Catalan Family, \( B \), we have shown that the set of non-intersecting odd-even arcs is counted by the Catalan numbers.
Reflexive and Symmetric Relations on $[n]$

Alexandra Boldin

1 Introduction

A relation $R$ on $[n]$, denoted $iRj$ with $i, j \in [n]$, is any subset of $[n] \times [n]$. Let $\mathcal{R}_n$ be the set of relations on $[n]$ which satisfy the following conditions:

(1) $R$ is reflexive,

(2) $R$ is symmetric,

(3) If $1 \leq i < j < k \leq n$ and $iRk$, then $iRj$ and $jRk$.

A relation $R$ on the set $X$ is reflexive if and only if $xRx$ for all $x \in X$. $R$ is symmetric if and only if $xRy$ implies $yRx$ for all $x, y \in X$. We show that $R_n = |\mathcal{R}_n|$ is equal to the $n$th Catalan number, $C_n$.

2 Enumeration for $n \leq 4$

We denote ordered pairs $(i, j)$ as $ij$ and omit the pairs $ii$.

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Table 1: Enumeration of Relations on \([n]\) for \(n \leq 4\) satisfying the given conditions. We do not list the pairs \(ii\) that also appear in each relation.

### 3 Bijection to Dyck Paths of length \(2n\)

We construct a bijection from this problem to a known Catalan family. We define the function \(f\) that maps the relations described to Dyck paths of length \(2n\). The set of Dyck paths, or mountains, is denoted by \(\mathcal{M}_n\). Thus \(f : \mathcal{R}_n \to \mathcal{M}_n\).

Condition (3) is essential for the mapping \(f\). It ensures that for any point \((x, y)\) on the coordinate plane, each point between it and the line \(y = x\) is also included in the relation. More specifically, if we connect the point \((x, y)\) to the line \(y = x\) with a horizontal and vertical line, each integer coordinate on these lines as well as any integer coordinates between these lines and the line \(y = x\) are included in the relation. For example, if the point \((1,5)\) were included in the relation, condition 3 implies that \((1,2), (2,5), (1,3), (3,5), (1,4), (4,5), (2,3), (3,4),\) and \((2,4)\) are all also included in the relation. This set of points is pictured below, with point \((1,5)\) larger than the rest for emphasis.
The implication of this condition on the mapping is clear, all points will form triangular configurations above the line $y = x$. Thus, this condition ensures that the points will map clearly to mountain ranges.

### 3.1 Mapping

We begin by plotting each ordered pair in the relation on the coordinate plane. For example, the relation \{23, 32\} is plotted as:

![Plot of (2, 3) and (3, 2)]

Next we divide the plot along the line $x = y$. Because the relation $R$ is symmetric, the plot will always be symmetric over this line. We disregard everything below this line. We then connect all points to any other point that is one unit away. We also connect any points within one unit of the line $x = y$ to the line. Because of condition (3), there will always be at least one point within one unit of the line $x = y$. The connecting lines are shown below:

![Connecting lines to (2, 3) and (3, 2)]
Next, we highlight any portions of the line $x = y$ between $(1, 1)$ and $(n, n)$ that are not underneath a blue triangle.

![Diagram showing highlighted portions between $(1, 1)$ and $(n, n)$]

We add a triangle below each diagonal segment.

![Diagram showing added triangles]

Finally, we add a vertical line from $(1, 0)$ to $(1, 1)$ and a horizontal line from $(n, n)$ to $(n + 1, n)$.

![Diagram showing added line segments]

When we rotate this image clockwise $45^\circ$, the mapping to Dyck Paths becomes clear. Following the red, green, and blue lines yields the following image, a Dyck Path of length $2n$.

![Diagram of Dyck Path]

Below we show a larger example for the following relation on $[5]$:

$\{12, 21, 13, 31, 14, 41, 23, 32, 24, 42, 34, 43\}$
Figure 1: Steps showing the mapping from a relation on \([5]\) to a Dyck Path of length 10.

3.2 Bijection

As is clear from the images above, \(f\) does in fact map relations on \([n]\) to Dyck paths of length \(2n\). The farthest that any paths come below the line \(y = x\) is one step down. Since we add the first step up and final step down, this means that the mountain range never goes below sea level. Additionally, there will always be \(2n\) total steps because we add \(n\) steps up and \(n\) steps down. Thus \(f : R_n \rightarrow M_n\) is a well-defined function. We now show that this function is a bijection by proving that it is both surjective and injective.

3.2.1 Surjective

Every Dyck path is mapped to by a relation \(R_n\). To show this, we describe how to map a Dyck path to a relation on \([n]\). Begin by adding a horizontal line one step above the bottom of the path. Next, remove any paths below this line. Because this line cuts off only the lowest level of the mountain range, all paths below the line will be valleys of the form \((1, -1), (1, 1)\). Rotate the remaining paths counterclockwise by \(45^\circ\) and place on an \((n + 1) \times (n + 1)\) coordinate grid such that the previously drawn horizontal line marks \(y = x\). Place a point at all integer coordinates on the Dyck path and between the path and the line \(y = x\). Reflect these points over the line \(y = x\). We now have the list of points in the relation. We can easily add any points of the form \((i, i)\) to this relation so it satisfies condition (1) without changing the mapping from the mountain ranges. Because we reflect
all points over the line $y = x$, the relation is symmetric, satisfying condition (2). Finally, because we placed a point at each integer coordinate on or between the Dyck path and the line $y = x$, we know that this relation satisfies condition 3. Thus, the mapping $f : \mathcal{R}_n \rightarrow \mathcal{M}_n$ is surjective.

3.2.2 Injective

Condition 3 is also crucial to the argument that the mapping $f$ is injective. As described above, condition 3 requires that for any point in a relation, all points in the triangle between that point and the line $y = x$ are also included in the relation. To show that any additional point will create an entirely new peak, let $R \neq S$ be relations. Without loss of generality, there will be a point $(x, y) \in R \setminus S$. After the mapping, this point will be contained in a peak of $f(R)$ that does not appear in $f(S)$. This means that each distinct relation maps to a distinct mountain range. Thus, the mapping $f$ is injective.

4 Conclusion

We have described a bijection between the relations on $[n]$ satisfying the given conditions and Dyck Paths of length $2n$. Dyck Paths of length $2n$ are a known Catalan family, thus the number of relations on $[n]$ satisfying the described conditions is equal to the $n^{th}$ Catalan number.
Timber! Even Catalan Plays Games

Tom Wakin* and Ian Calaway†

1 Introduction

Timber! is an impartial combinatorial game played with $2n$ dominos arranged in a line. Half of these are black and the other half white. During a move, a player chooses a domino and topples it. Black dominos must be toppled to the left, and white dominos must be toppled to the right. When a player topples a domino, that domino knocks over all the following dominos in that direction, independent of their color. The toppled dominos are then removed. Whichever player makes the last valid move wins.

An example of gameplay is demonstrated below. The game begins with the set up below.

\[
\begin{array}{cccccccc}
\text{I} & \text{I} & \text{I} & \text{I} & \text{I} & \text{I} & \text{I} & \text{I} \\
\end{array}
\]

The first player, Alex, topples the fifth domino, toppling the rest to the left of it.

\[
\begin{array}{cccccccc}
\text{I} & \text{I} & \text{I} & \text{I} & \text{I} & \text{I} & \text{I} & \text{I} \\
\end{array}
\]

The second player, Boris, topples the third domino, which does not topple any other domino.

---

*Tom lives in Saint Paul, MN. He was raised in New York City and knows that Ian always lies.
†Ian lives in Saint Paul, MN. He was raised in Dubuque and knows that Tom always tells the truth.
Alex topples the last black domino.

From here, Boris has only one move to make, toppling the last domino, which results in his victory, as it is the last valid move.

We prove that the number of losing starting positions in the game of Timber! follows the Catalan recurrence. We prove this by defining a direct bijection between these losing positions and Dyck paths, a known Catalan family.

2 Game Theory

2.1 \( P \)-positions and \( N \)-positions

As in any impartial combinatorial game, every Timber! position is either a \( P \)-position or an \( N \)-position. An \( N \)-position describes a game position where the player who goes next is guaranteed to win, assuming optimal play. The set of \( N \)-positions is denoted by \( N \). A \( P \)-position describes a game position where the player who went previously is guaranteed to win, again assuming optimal play. The set of \( P \)-positions is denoted by \( P \).

In order to distinguish between \( P \)-positions and \( N \)-positions, we must look at the followers of a given position, where followers are the possible positions reached after a move.

It is known that \( p \in P \) if and only if all positions immediately following \( p \) are \( N \)-positions. Also, \( n \in N \) if and only if \( n \) is followed by at least one \( P \)-position. An analysis of possible starting positions for \( n = 2 \) will give insight to the differences between \( P \)-positions and \( N \)-positions.
2.2  $n = 2$ Games

First, we consider the following starting position with 2 black dominoes and 2 white dominoes.

\[
\begin{array}{cccc}
\text{a} & \text{b} & \text{c} & \text{d} \\
\end{array}
\]

The immediate followers of this game are:

\[
\begin{array}{cccc}
\text{a} & \text{b} & \text{c} & \text{d} \\
\end{array}
\]

(a)  (b)  (c)  (d)

Position (a) is clearly an $N$-position as the next player can topple the rightmost domino and finish the game. Position (b) is also an $N$-position, although it is less obvious. For position (b), the next player to make a move can topple the first domino, leaving a black domino and white domino, an obvious $P$-position. Position (c) is another $N$-position since the next player can topple the rightmost domino and finish the game. Finally, position (d) is an $N$-position since the next player can topple the rightmost domino leaving a black and white domino, which is a $P$-position. Note that positions (a) and (c) are symmetric, as are (b) and (d). Because all followers of the game position $\begin{array}{cccc}
\text{a} & \text{b} & \text{c} & \text{d} \\
\end{array}$ are elements of set $N$, this arrangement is a $P$-position.

Here is a second possible starting position:

\[
\begin{array}{cccc}
\text{a} & \text{b} & \text{c} & \text{d} \\
\end{array}
\]

The immediate followers of this game are:

\[
\begin{array}{cccc}
\text{a} & \text{b} & \text{c} & \text{d} \\
\end{array}
\]

(a)  (b)  (c)  (d)
These followers are all $N$-positions. For followers (a) and (d), the next player simply has to topple the rightmost black domino, and for followers (b) and (c), the next play can topple the leftmost white domino to finish the game. Since these are $N$-positions, the starting position is a $P$-position.

Next, we consider the following starting position.

\[
\begin{array}{ccc}
| & | & |
\end{array}
\]

The immediate followers of this game are:

\[
\begin{array}{ccc}
| & | & | & | \\
| & | & | \\
| & | & | & | \\
\end{array}
\]

(a) (b) (c) (d)

The followers of this game include the $P$-position (d), therefore this starting position must be an $N$-position.

Here are the rest of the starting positions.

\[
\begin{array}{ccc}
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
\end{array}
\]

The other arrangements are those that begin with a White domino. All of these are $N$-positions because the first person could just topple that leftmost white domino, ending the game.

2.3 $P$-positions vs $N$-positions

The games in Section 2.2 reveal the difference between $P$-positions and $N$-positions. For a $P$-position in Timber!,

1. When reading dominos from left to right, the number of black dominos must be greater than or equal to the number of white dominoes.
2. When reading the dominos from right to left, the number of white dominos must be greater than or equal to the number of black dominos.
3. The total number of black dominos equals the total number of white dominos.

We will denote the set of positions that satisfy these conditions as $\mathcal{A}$, and those that do not satisfy these conditions as $\mathcal{B}$. We will show that if $p \in \mathcal{A}$ then all of its followers are in $\mathcal{B}$, and that if $p \in \mathcal{B}$ then $p$ has at least one follower that is in $\mathcal{A}$. This will prove that $\mathcal{A}$ is the set of $P$-positions and $\mathcal{B}$ is the set of $N$-positions.

Given a position in $\mathcal{A}$, we show that any follower will have more dominos of one color than the other. This follower violates condition (3) above, so it belongs to $\mathcal{B}$. Let us describe a position in $\mathcal{A}$ as $x_1, x_2, \ldots, x_{2n}$ where each $x_i$ represents a black or white domino. If we pick a black domino $x_k$, then $x_1, \ldots, x_{k-1}$ has at least as many black dominos as white dominos. This means that $x_{k+1}, \ldots, x_{2k}$ has more white dominos than black dominos. By toppling the domino $x_k$, we are left with $x_{k+1}, \ldots, x_{2k}$. Since there are now more white dominos than black dominos, the position is in $\mathcal{B}$. The argument is symmetric when $x_k$ is a white domino.

Next, we prove that each position in $\mathcal{B}$ has at least one follower in $\mathcal{A}$. Let us describe a position in $\mathcal{B}$ as $x_1, x_2, \ldots, x_{2n}$ where each $x_i$ again represents a black or white domino. We will let $x_k$ be the first white domino, when reading left to right, such that $x_1, \ldots, x_{k-1}$ has an equal number of black and white dominos. This definition of $x_k$ also guarantees that $x_1, \ldots, x_{k-1}$ meets all the conditions of a position in $\mathcal{A}$. This means we can topple $x_k$, and be left with a position in $\mathcal{A}$. The argument is again symmetric when $x_k$ is the first black domino, when reading right to left, such that $x_{k+1}, \ldots, x_{2n}$ has an equal number of black and white dominos. Therefore, for every position in $\mathcal{B}$, there exists a follower in $\mathcal{A}$.

For an example of this property, let us consider this position in $\mathcal{A}$.

```
|   |   |   |   |   |   |   |
```

Alex does not have an optimal move, so he topples the $8^{th}$ domino right. This is the resulting position.

```
|   |   |   |   |   |   |   |
```

Now, when reading right-to-left, there are more black dominos than white dominos. This ordering contradicts the conditions for an $A$ position, placing it in the set of $B$ positions. The domino that acts as the contradiction is the $3^{rd}$. Boris now goes and topples the $3^{rd}$ domino. This is the resulting position.

\[ 
\begin{array}{c}
\text{I} \\
\text{II} \\
\text{III} \\
\text{IV} \\
\end{array}
\]

This position satisfies the conditions for an $A$ position.

We have shown that all followers of a position in $A$ are in $B$, and that each position in $B$ has at least one follower in $A$. Thus we have proven, $A = \mathcal{P}$ and $B = \mathcal{N}$.

\section{Catalan Recurrence Relation}

We give a bijection between the set $A_n$ of $\mathcal{P}$-positions with $2n$ dominoes and the set $D_n$ of Dyck paths of length $2n$. Define $f : A_n \to D_n$ as follows. We map black dominoes to up slopes and white dominoes to down slopes. In each $a \in A_n$, there are $n$ black dominos and $n$ white dominos, just as there are $n$ up slopes and $n$ down slopes in each Dyck path. Additionally, in a $\mathcal{P}$-position of Timber!, the arrangement must always start with a black domino, just like a Dyck path must always start with an up slope. The arrangement must end in a white domino, just as Dyck paths must always end in down slopes. Furthermore, when reading the arrangement of the dominos from left to right there are at least as many black dominoes as white dominoes, if not more black than white. Likewise, when reading Dyck paths from left to right, there are at least as many upslopes as downslopes. Table 1 shows this mapping for $0 \leq n \leq 4$. 
We now show that $f$ is a bijection. As shown in Table 1, each arrangement corresponds to a unique path. The inverse function $f$ has an inverse, $F^{-1}$, that maps does the opposite. The inverse function is defined as mapping is clear: we map up and down slopes to black and white dominos, respectively. Therefore, the $\mathcal{P}$-positions of Timber! follow the Catalan recurrence.

### 4 Acknowledgment

We thank Richard Nowakowski for sharing this result with us.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Dyck paths of length $2n$</th>
<th>$\emptyset$</th>
</tr>
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<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
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<td></td>
<td>$\wedge$</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>$\wedge$</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>$\wedge$</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>$\wedge$</td>
</tr>
</tbody>
</table>

Table 1: An enumeration of Timber! $\mathcal{P}$-positions and their corresponding Dyck paths.