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Preface

This book will not show you how to do calculus. My intent is instead to explain how and why it arose. Too often, its narrative structure is lost, disappearing behind rules and procedures. My hope is that readers of this book will find inspiration in its story. I assume some knowledge of the tools of calculus though, in truth, most of what I have written requires little more than mathematical curiosity.

Most of those who have studied calculus know that Newton and Leibniz "stood on the shoulders of giants" and that the curriculum we use today is not what they handed down over 300 years ago. Nevertheless, it is disturbingly common to hear this subject explained as if it emerged fully formed in the late 17th century and has changed little since. The fact is that the curriculum as we know it today was shaped over the course of the 19th century, structured to meet the needs of research mathematicians. The progression we commonly use today and that AP® Calculus has identified as the *Four Big Ideas* of calculus: *limits*, *derivatives*, *integrals*, and finally *series*, is appropriate for a course of analysis that seeks to understand all that can go wrong in attempting to use calculus, but it presents a difficult route into *understanding* calculus. The intent of this book is to use the historical development of these four big ideas to suggest more natural and intuitive routes into calculus.

The historical progression began with integration, or, more properly, accumulation. This can be traced back at least as far as the 4th century BCE, to the earliest explanation of why the area of a circle is equal to that of a triangle whose base is the circumference of the circle ($\pi \times$ diameter) and whose height is the radius.[1] In the ensuing centuries, Hellenistic philosophers became adept at deriving formulas for areas and volumes by imagining geometric objects as built from thin slices. As we will see, this approach was developed further by Islamic, Indian, and Chinese philosophers, reaching its apex in 17th century Europe.

Accumulation is more than areas and volumes. In 14th century Europe, philosophers studied variable velocity as the rate at which distance is changing at each instant. Here we find the first explicit use of accumulating small changes in distance to find the total distance that is travelled. These philosophers realized that if the velocity is represented by distance above a horizontal axis, then the area between the curve representing velocity and the horizontal axis corresponds to the distance that has been travelled. Thus, accumulation of distances can be represented as an accumulation of area, connecting geometry to motion.

[1] In modern notation, $\frac{1}{2} \times (\pi \times 2r) \times r$ or πr^2.

The next big idea to emerge was differentiation, a collection of problem-solving techniques whose core idea is ratios of change. Linear functions are special because the ratio of the change in the output to the change in the input is constant. In the middle of the first millennium of the Common Era, Indian astronomers discovered what today we think of as the derivatives of the sine and cosine as they explored how changes in arc length affected changes in the corresponding lengths of chords. They were exploring *sensitivity*, one of the key applications of the derivative: understanding how small changes in one variable will affect another variable to which it is linked.

In 17th century Europe, the study of ratios of change appeared in the guise of tangent lines. Eventually, these were connected to the general study of rates of change. Calculus was born when Newton, and then independently Leibniz, came to realize that the techniques for solving problems of accumulation and ratios of change were inverse to each other, thus enabling natural philosophers to use solutions found in one realm to answer questions in the other.

The third big idea to emerge was that of series. Though written as infinite summations, infinite series are really limits of sequences of partial sums. They arose independently in India around the 13th century and Europe in the 17th, building from a foundation of the search for polynomial approximations. By the time calculus was well established, in the early 18th century, series had become indispensable tools for the modeling of dynamical systems, so central that Euler, the scientist who shaped 18th century mathematics and established the power of calculus, asserted that any study of calculus must begin with the study of infinite summations.

The term *infinite summation* is an oxymoron. "Infinite" literally means without end. "Summation," related to "summit," implies bringing to a conclusion. An infinite summation is an unending process that is brought to a conclusion. Applied without care, it can lead to false conclusions and apparent contradictions. It was largely the difficulties of understanding these infinite summations that led, in the 19th century, to the development of the last of our big ideas, the limit. The common use of the word "limit" is loaded with connotations that easily lead students astray. As Grabiner has documented,[2] the modern meaning of limits arose from the algebra of inequalities, inequalities that bound the variation in the output variable by controlling the input.

The four big ideas of calculus in their historical order, and therefore our chapter headings, are

(1) Accumulation (Integration)
(2) Ratios of Change (Differentiation)
(3) Sequences of Partial Sums (Series)
(4) Algebra of Inequalities (Limits).

In addition, I have added a chapter on some aspects of 19th century analysis. Just as no one should teach algebra who is ignorant of how it is used in calculus, so no one should teach calculus who has no idea how it evolved in the 19th century. While strict adherence to this

[2](Grabiner, 1981)

historical order may not be necessary, anyone who teaches calculus must be conscious of the dangers inherent in departing from it.

How did we wind up with a sequence that is close to the reverse of the historical order: limits first, then differentiation, integration, and finally series? The answer lies in the needs of the research mathematicians of the 19th century who uncovered apparent contradictions within calculus. As set out by Euclid and now accepted as the mathematical norm, a logically rigorous explanation begins with precise definitions and statements of the assumptions (known in the mathematical lexicon as *axioms*). From there, one builds the argument, starting with immediate consequences of the definitions and axioms, then incorporating these as the building blocks of ever more complex propositions and theorems. The beauty of this approach is that it facilitates the checking of any mathematical argument.

This is the structure that dictated the current calculus syllabus. The justifications that were developed for both differentiation and integration rested on concepts of limits, so logically they should come first. In some sense, it now does not matter whether differentiation or integration comes next, but the limit definition of differentiation is simpler than that of accumulation, whose precise explication as set by Bernhard Riemann in 1854 entails a complicated use of limit. For this reason, differentiation almost always follows immediately after limits. The series encountered in first-year calculus are, for all practical purposes, Taylor series, extensions of polynomial approximations that are defined in terms of derivatives. As used in first-year calculus, they could come before integration, but the relative importance of these ideas usually pushes integration before series.

The progression we now use is appropriate for the student who wants to verify that calculus is logically sound. However, that describes very few students in first-year calculus. By emphasizing the historical progression of calculus, students have a context for understanding how these big ideas developed.

Things would not be so bad if the current syllabus were pedagogically sound. Unfortunately, it is not. Beginning with limits, the most sophisticated and difficult of the four big ideas, means that most students never appreciate their true meaning. Limits are either reduced to an intuitive notion with some validity but one that can lead to many incorrect assumptions, or their study devolves into a collection of techniques that must be memorized.

The next pedagogical problem is that integration, now following differentiation, is quickly reduced to antidifferentiation. Riemann's definition of the integral—a product of the late 19th century that arose in response to the question of how discontinuous a function could be yet still be integrable—is difficult to comprehend, leading students to ignore the integral as a limit and focus on the integral as antiderivative. Accumulation is an intuitively simple idea. There is a reason this was the first piece of calculus to be developed. But students who think of integration as primarily reversing differentiation often have trouble making the connection to problems of accumulation.

The current curriculum is so ingrained that I hold little hope that this book will cause everyone to reorder their syllabi. My desire is that teachers and students will draw on the historical record to focus on the algebra of inequalities when studying limits, ratios of change when studying differentiation, accumulation when studying integration, and sequences of partial sums when studying series. To aid in this, I have included an appendix of practical insights and suggestions from research in mathematics education. I hope that this book will help teachers recognize the conceptual difficulties inherent in the definitions and theorems that were formulated in the 19th century and incorporated into the curriculum during the 20th. These include the precise definitions of limits, continuity, and convergence. Great mathematicians did great work without them. This is not to say that they are unimportant. But they entered the world of calculus late because they illuminate subtle points that the mathematical community was slow to understand. We should not be surprised if beginning students also fail to grasp their importance.

I also want to say a word about how I refer to the people involved in the creation of calculus. Before 1700, I refer to them as "philosophers" because that is how they thought of themselves, as "lovers of wisdom" in all its forms. None restricted themselves purely to the study of mathematics. Newton and Leibniz are in this company. Newton referred to physics as "natural philosophy," the study of nature. From 1700 to 1850, I refer to them as "scientists." Although that word would not be invented until 1834, it accurately captures the broad interests of all those who worked to develop calculus during this period. Many still considered themselves to be philosophers, but the emphasis had shifted to a more practical exploration of the world around us. Almost all of them included an interest in astronomy and what today we would call "physics." After 1850, it became common to focus exclusively on questions of mathematics. In this period and only in this period, I will refer to them as mathematicians.

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