Hints, Comments, and Solutions for Exercises in *Proofs and Confirmations*

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This is a work in progress. Comments and corrections are welcome and will be added with grateful attribution.

1.1.5 This turned out to be much more difficult than I anticipated. It is easy to see that any subset that contains all permutation matrices cannot contain any ASM's with any -1's (otherwise, we can permute the columns so that we get a -1 in the first column). I believe that given any ASM with at least one -1, some power of it must fail to be an ASM. Heuristically, taking powers increases the number of -1s and/or moves the -1s closer to the boundary, but I would very much appreciate a simple proof that any ASM with a -1 in it has a power that fails to be an ASM.

1.1.12 The conjecture is that $A_{n,k}$ can be odd if and only if $n = (4^t - 1)/3$, $(4^t + 2)/3$, $(2 \cdot 4^t + 1)/3$ or $(2 \cdot 4^t + 4)/3$ for some integer $t \geq 1$. If $n = (4^t - 1)/3$, then $A_{n,k}$ is odd if and only if $k = (n-1)/2, (n+1)/2$, or $(n+3)/2$. If $n = (2 \cdot 4^t + 1)/3$, then $A_{n,k}$ is odd if and only if $k = (n+1)/2$. Let $A(t)$ (respectively, $B(t)$) be the set of values of k for which $A_{n,k}$ is odd when $n = (4^t + 2)/3$ (respectively, $n = 2(4^t + 2)/3$). Then $A(t) = \{k \,|\, k \in A(t-1) \text{ or } n+1-k \in A(t-1)\}$ (respectively, $B(t) = \{k \,|\, k \in B(t-1) \text{ or } n+1-k \in B(t-1)\}$).

n	k
1	1
2	1, 2
3	2
4	1, 4
5	2, 3, 4
6	1, 2, 5, 6
11	6
12	1, 4, 9, 12
21	10, 11, 12
22	1, 2, 5, 6, 17, 18, 21, 22
43	22
44	1, 4, 9, 12, 33, 36, 41, 44
85	42, 43, 44
86	1, 2, 5, 6, 17, 18, 21, 22, 65, 66, 69, 70, 81, 82, 85, 86
171	86
172	1, 4, 9, 12, 33, 36, 41, 44, 129, 132, 137, 140, 161, 164, 169, 172

1.1.13 I do not have a proof for this.

1.1.14 I do not know of a slick proof. It can be forced through by considering the three cases that depend on the residue class of n modulo 3, simplifying the rational product, and then demonstrating that an arbitrary prime power divides the numerator at least as often as it divides the denominator.

1.2.2 All terms on the right cancel except those with $k = 1$ in the denominator and with $k = t$ in the numerator.

1.2.3 1085787; 90379784; 267227532

1.2.5 r orbits of size 1, no orbits of size 2, $r(r-1)$ orbits of size 3, $\binom{r}{3}$ orbits of size 6.

1.2.6 r orbits of size 1, no orbits of size 2, $(r^3 - r)/3$ of size 3, no orbits of size 6.

1.2.9 1185; 151008

1.2.10 The product in question is equal to

$$\begin{aligned}
&\frac{(1-q^2)(1-q^5)(1-q^8)(1-q^{3\cdot3})(1-q^{3\cdot4})}{(1-q)(1-q^4)(1-q^7)(1-q^{3\cdot2})(1-q^{3\cdot3})} \\
&\qquad\times \frac{(1-q^{3\cdot4})(1-q^{3\cdot6})(1-q^{3\cdot6})(1-q^{3\cdot7})(1-q^{6\cdot5})}{(1-q^{3\cdot3})(1-q^{3\cdot5})(1-q^{3\cdot5})(1-q^{3\cdot6})(1-q^{6\cdot4})} \\
&\qquad\quad = \frac{(1+q)(1+q^6)(1+q^9)(1+q^{15})(1+q^7+q^{14})}{(1-q^4+q^8)(1+q^5+q^{10})}.
\end{aligned}$$

1.2.11 The orbits of $\mathcal{B}(r,r,r)/\mathcal{S}_3$ that are not in $\mathcal{B}(r-1,r-1,r-1)/\mathcal{S}_3$ correspond to lattice points (r,i,j) with $1 \le i \le j \le r$:

$$\frac{\mathtt{TSPP}(r)}{\mathtt{TSPP}(r-1)} = \prod_{i=1}^{r}\prod_{j=i}^{r} \frac{i+j+r-1}{i+j+r-2} = \prod_{i=1}^{r} \frac{i+2r-1}{2i+r-2}.$$

1.2.12

$$\emptyset, 1, \begin{matrix} 2 & 1 \\ 1 & \end{matrix}, \begin{matrix} 2 & 2 \\ 2 & 1 \end{matrix}, \begin{matrix} 2 & 2 \\ 2 & 2 \end{matrix}.$$

1.3.1 (a)

$$\begin{matrix} 4 & 4 & 3 & 2 \\ 4 & 4 & 3 & 2 \\ 3 & 3 & 3 & \\ 2 & 2 & & \end{matrix}$$

(b)
$$\begin{array}{ccccc} 5 & 5 & 4 & 3 & 3 \\ 5 & 5 & 4 & 3 & 2 \\ 5 & 4 & 4 & 3 & \\ 3 & 3 & 3 & & \\ 2 & 2 & 1 & & \end{array}$$

1.3.2 (a)
$$\begin{array}{ccc} 3 & 3 & 2 \\ & 2 & 1 \end{array}$$

(b)
$$\begin{array}{ccccc} 5 & 4 & 3 & 3 & 3 \\ & 3 & 2 & 1 & \\ & & 1 & & \end{array}$$

1.3.3 (a) 30; (b) 19 + 3(6+5+4+3+3+2) + 16 + 3(3+3+2+1+1) + 7 + 3(2+1) + 4 = 154

1.3.4 $\sum_j \left(3a_{j,j} - 2 + 3(a_{j,j+1} + \cdots + a_{j,r_j} - r_j + j)\right)$

1.3.5 Hint: Use the fact that $1 - q^a = (1 - q^j) + q^j(1 - q^{a-j})$.

1.3.9
$$\prod_{1 \le i \le j \le r} \frac{r+i+j-1}{2i+j-1} = \prod_{j=1}^{r} \frac{(r+2j-1)!}{(r+j-1)!} \prod_{i=1}^{r} \frac{(3i-2)!}{(2i+r-1)!}.$$

1.3.11 This is nonsense. Every descending plane partition satisfies the definition of being stripped.

2.1.2 5; 7

2.1.3 It is equal to the number of partitions of n into at most two parts. The larger of these two parts can be any integer that is greater than or equal to $n/2$, and there are $1 + \lfloor n/2 \rfloor$ possibilities.

2.1.4 This is the number of partitions of n into at most three parts which is the integer closest to $(n+3)^2/12$. We have that
$$\frac{1}{(1-q)(1-q^2)(1-q^3)} = \frac{1/6}{(1-q)^3} + \frac{1/4}{(1-q)^2} + \frac{1/4}{(1-q^2)} + \frac{1/3}{(1-q^3)}.$$
Comparing coefficients of q^n, we see that the desired number is equal to
$$\frac{1}{6}\binom{n+2}{2} + \frac{1}{4}\binom{n+1}{1} + \frac{1}{4}\chi(2|n) + \frac{1}{3}\chi(3|n).$$
The general problem of the number of partitions of n into at most k parts is discussed in example 2, chapter 5 and example 1, chapter 12 of Andrews, *The Theory of Partitions*.

2.1.9 $3m - 2$

2.1.10 First assume both signs are the same, then we can write the equality as

$$3(j^2 - k^2) = \mp(j - k).$$

If $j \neq k$, then we can divide both sides by $j - k$ to get that $3(j + k) = \mp 1$. This cannot be since three divides the left side but not the right. If the signs are different, we can write the equality as $3(j^2 - k^2) = \pm(j + k)$. Since j and k are positive, $j + k$ is not zero and so we can divide by it: $3(j - k) = \pm 1$. again, this cannot be.

2.1.17

$$\prod_{i=1}^{\infty} \frac{1 - q^{d \cdot i}}{1 - q^i} = \prod_{i=1}^{\infty} \left(1 + q^i + q^{2i} + \cdots + q^{(d-1)i}\right).$$

2.1.18 This is Fabian Franklin's proof which he discovered while in Sylvester's class on partition theory. Given a partition $\lambda = (\lambda_1 > \lambda_2 > \cdots \geq \lambda_r > 0)$ into distinct parts, let $\sigma(\lambda)$ be the largest integer such that $\lambda_{i-1} - \lambda_i = 1$ for $i \leq \sigma$. If $\lambda_1 - \lambda_2 > 1$, then $\sigma = 1$. If $\lambda_r \leq \sigma$, then we delete the part λ_r and add one to each of the r largest parts. If $\lambda_r > \sigma$, then we subtract one from each of the *sigma* largest parts and add a new part of size σ. This is an involution on the set of partitions of n into distinct parts that changes the parity of the number of parts. The only time it cannot be applied is when $\sigma = r = \lambda_r$ or $\sigma = r = \lambda_r - 1$. The former is a partition of $r^2 + r(r-1)/2$ and the latter is a partition of $r^2 + r + r(r-1)/2$.

2.2.2 $\delta(\lambda)$ is the size of the largest square that will fit inside the partition; the respective values are 3, 3, and k. $\sigma(\lambda)$ is the number of consecutive parts, starting with the largest, that differ by exactly 1; the respective values are 2, 2, and k.

2.2.3 17 + 17 + 15 + 1

2.2.7 Hint: In the partition with no multiples of d, write the number of times each integer appears as a sum of powers of d mutliplied by coefficients that are less than or equal to $d - 1$.

2.2.8 Add one to each of the parts in the second partition, subtract one from each part in the first partition, and then interchange the two partitions.

2.2.9 If there is a 0 in the second partition, we remove it and then interchange the partitions. If there is no 0 in the second partition, we append a 0 to the first partition and then interchange them.

2.2.10 This is the Jacobi triple product identity with q replaced by q^4 and then $x = -q^{-1}$.

2.2.11 This is the Jacobi triple product identity with q replaced by q^m and then $x = -q^{-a}$.

2.2.12

$$\begin{aligned} g(x/q) &= g(x)\frac{(1-x)(1-x^2q^{-1})}{(1-x^{-1})(1-x^{-2}q)} = g(x)x^3q^{-1}. \\ g(x^{-1}) &= g(x)\frac{1-x}{1-x^{-1}} = g(x)(-x). \end{aligned}$$

2.2.13 Using the first equality and then the second: $b_1 = b_{-2} = -b_1$. From the second equality: $b_{-1} = -b_0$.

2.2.15 Equation (2.13) simplifies to

$$-2q^{-2}\prod_{k=1}^{\infty}(1+q^k)(1-q^{2k-1}) = -2q^{-2}b_0\prod_{k=1}^{\infty}(1-q^k).$$

2.3.2 No.

2.3.4 Taking the derivative of the generating function, we get

$$\begin{aligned} \sum_{n=1}^{\infty} p_{\mathcal{O}}(n)\, q^{n-1} &= \prod_{m=1}^{\infty}\frac{1}{1-q^{2m-1}}\sum_{m=1}^{\infty}\frac{(2m-1)q^{2m-2}}{1-q^{2m-1}} \\ &= \prod_{m=1}^{\infty}\frac{1}{1-q^{2m-1}}\sum_{m=1}^{\infty}\sum_{t\geq 0} q^{(2m-1)(t+1)-1} \\ &= q^{-1}\left(\sum_{n=1}^{\infty} p_{\mathcal{O}}(n)\, q^n\right)\left(\sum_{d=1}^{\infty}\sigma_{\text{odd}}(d)\, q^d\right). \end{aligned}$$

2.3.6 Exercises 2.3.4 and 2.3.5 imply that

$$\begin{aligned} \sigma_{\text{odd}}(n) &= np_{\mathcal{O}}(n) - \sum_{j=1}^{n-1}\sigma_{\text{odd}}(j)p_{\mathcal{O}}(n-j), \\ \sigma_{\text{alt}}(n) &= np_{\mathcal{D}}(n) - \sum_{j=1}^{n-1}\sigma_{\text{alt}}(j)p_{\mathcal{D}}(n-j). \end{aligned}$$

The proof is then by induction, using the fact that $p_{\mathcal{O}}(n) = p_{\mathcal{D}}(n)$.

To prove this directly, write $j = 2^a \cdot i$ where i is odd, then $\sigma_{\text{odd}}(j) = \sigma(i)$. On the other hand

$$\begin{aligned} \sigma_{\text{alt}}(j) &= \sum_{k|i}\sum_{r=0}^{a} 2^r k(-1)^{1+2^{a-r}/k} \\ &= \left(\sum_{r=0}^{a-1} -2^r\sigma(i)\right) + 2^a\sigma(i) \\ &= \sigma(i). \end{aligned}$$

2.3.7

$$\sum_{n=1}^{\infty} n\, a_k(n)x^n = k\left(\sum_{n=1}^{\infty} a_k(n)x^n\right)\left(\frac{1}{1-x} - \frac{6x^6}{1-x^6}\right).$$

We have that $a_k(n) = 0$ for $n < k$, $a_k(k) = 1$, and for $n > k$:

$$a_k(n) = \frac{k}{n-k}\sum_{j=1}^{n-k} b_j\, a_k(n-j)$$

where $b_j = 1$ if 6 does not divide j and $b_j = -5$ if 6 divides j.

2.3.11 Given the power series $1 + \sum_{r=1}^{\infty} b_r q^r$, let $a_1 = b_1$. The coefficient of q^0 and q^1 agree in the power series with those in the expansion of $(1-q)^{-b_1}$. Assume that we have found unique integers $a_1, \ldots, a_{j-1}$ such that the coefficients of $q^0, q^1, \ldots, q^{j-1}$ in the expansion of $\prod_{i=1}^{j-1}(1-q^i)^{-a_i}$ agree with those in the power series, and let c_j be the coefficient of q^j in the expansion of this product. the coefficient c_j is an integer. Let $a_j = b_j - c_j$. Check that the coefficient of q^j in $\prod_{i=1}^{j}(1-q^i)^{-a_i}$ is equal to b_j.

2.3.14 (a) $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

2.3.15 It is the sum over all rows, except the bottom row, of the number of integers in that row that do not appear in the row below.

2.3.18 $MT_n(\{1, i\}) = MT_{n-1}(i)$.

2.4.7

$$\det\left(\binom{x_i + j}{j}\right) = \prod_{i=1}^{n} \frac{x_i + 1}{i^{n+1-i}} \det\left((x_i+2)(x_i+3)\cdots(x_i+j) \right).$$

The determinant on the right is an alternating polynomial in the x_i of total degree $n(n-1)/2$ and so it is equal to the Vandermonde product times a constant which must be $(-1)^{\binom{n}{2}}$.

$$\det\left(\binom{x_i + j}{j}\right) = (-1)^{\binom{n}{2}} \prod_{i=1}^{n} \frac{x_i + 1}{i^{n+1-i}} \prod_{1 \le i < j \le n} (x_i - x_j).$$

3.1.1 Conjugate the Ferrers graph.

3.1.3 Using the recursive formula, we have that

$$\begin{aligned} f(q,m) &= 1-\begin{bmatrix} m-1 \\ 1 \end{bmatrix} - q^{m-1}\begin{bmatrix} m-1 \\ 0 \end{bmatrix} \\ &+ \begin{bmatrix} m-1 \\ 2 \end{bmatrix} + q^{m-2}\begin{bmatrix} m-1 \\ 1 \end{bmatrix} - \cdots + (-1)^m q^0 \begin{bmatrix} m-2 \\ m-2 \end{bmatrix} \\ &= \sum_{j=1}^{m} (-1)^{j-1}(1-q^{m-j})\begin{bmatrix} m-1 \\ m-j \end{bmatrix} \\ &= \sum_{j=1}^{m} (-1)^{j-1}(1-q^{m-1})\begin{bmatrix} m-2 \\ j-1 \end{bmatrix} \\ &= (1-q^{m-1})f(q,m-2). \end{aligned}$$

3.1.4 Note that $\alpha^k = 1$.

3.1.5 $\alpha^{j^2+j} = \alpha^{j^2+j(k+1)}$ amd the sum in $G(\alpha)$ can be taken over any set of residues modulo k.

3.1.6 $\alpha^{[(k+1)/2]^2} = \alpha^{(k-1)^2/4} = \alpha^{1+3+\cdots+k-2}$.

3.1.10 Note that the conditions should read "positive for $k \equiv \pm 1 \pmod 8$ and negative for $k \equiv \pm 3 \pmod 8$." The sign of the product is positive if there are an even number of integers j in the interval $(k/4, (k-1)/2]$ and negative if there are an odd number of integers in this interval.

3.1.13 For $j = 2$, this identity is

$$\begin{aligned} \begin{bmatrix} n+2 \\ 2 \end{bmatrix} &= q^2 \begin{bmatrix} n \\ 2 \end{bmatrix}\begin{bmatrix} 2n \\ 0 \end{bmatrix} + \begin{bmatrix} n \\ 0 \end{bmatrix}\begin{bmatrix} 2n-1 \\ 1 \end{bmatrix}, \\ \frac{(1-q^{n+1})(1-q^{n+2})}{(1-q)(1-q^2)} &= q^2\,\frac{(1-q^n)(1-q^{n-1})}{(1-q)(1-q^2)} + \frac{(1-q^{2n-1})}{(1-q)}. \end{aligned}$$

3.1.19 The coefficient of x^i is the generating function for partitions into exactly i parts, all of which are less than or equal to n. If we subtract 1 from each of these parts, we are left with the generating function for partitions into at most i parts, each of which is $\le n-1$, and this generating function is $\begin{bmatrix} n+i-1 \\ i \end{bmatrix}$. It follows that

$$\frac{1}{(xq;q)_n} = \sum_{i=0}^{n} \begin{bmatrix} n+i-1 \\ i \end{bmatrix} q^i x^i.$$

3.1.20

$$\prod_{j=1}^{i}(1-q^{n-i+j}) = \prod_{j=1}^{i}(-q^{n-i+j})(1-q^{-n+i-j}).$$

It follows that

$$\begin{bmatrix} n \\ i \end{bmatrix} q^{i(i-1)/2}(-x)^i = \frac{(q^{n-i+1};q)_i}{(q:q)_i} q^{i(i-1)/2}(-x)^i = \frac{(q^{-n};q)_i}{(q;q)_i}(xq^n)^i.$$

Note that if $i > n$, then one of the factors in $(q^{-n};q)_i$ is $1-1=0$.

3.1.21

$$\lim_{n\to\infty} \begin{bmatrix} n \\ i \end{bmatrix} = \lim_{n\to\infty} \frac{(q^{n-i+1};q)_i}{(q;q)_i} = \frac{1}{(q;q)_i}.$$

3.2.1 $\mathrm{maj}(11001010100) = 2+5+7+9 = 23$. $\mathcal{I}(01101011001) = 2+3+5+5 = 15$. $\mathcal{I}(11001010100) = 2+2+3+4+5+5 = 21$.

3.2.4 We proceed by induction on the number of variables. The previous exe4rcise established this result for $k=2$. Assume that it is correct up $k-1$, $k \geq 3$. Let σ be a sequence with m_i is, $1 \leq i < k$, and let λ be a partition into at most m_k parts less than or equal to $M = m_1 + \cdots = m_{k-1}$. We must establish a one-to-one correspondence between the set of pairs (σ, λ) and and the set of sequences, τ, with m_i is, $1 \leq i \leq k$, such that $\mathrm{maj}(\sigma) + |\lambda| = \mathrm{maj}(\tau)$. It then follows that

$$\begin{aligned} \sum_{\tau} q^{\mathrm{maj}(\tau)} &= \sum_{\sigma} q^{\mathrm{maj}(\sigma)} \sum_{\lambda} q^{|\lambda|} \\ &= \begin{bmatrix} m_1 + \cdots + m_{k-1} \\ m_1, \ldots, m_{k-1} \end{bmatrix} \begin{bmatrix} M + m_k \\ m_k \end{bmatrix} \\ &= \begin{bmatrix} m_1 + \cdots + m_k \\ m_1, \ldots, m_k \end{bmatrix}. \end{aligned}$$

We begin with the largest part in $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{m_k} \geq 0)$. If the entry in σ in position $\lambda_1 - 1$ is less than or equal to the next entry, then insert k after the entry in position $\lambda_1 - 1$. Otherwise, insert k in the first position to the left where it does not fall inside a pre-existing descent. Note that wherever we put this first k, we have increased the major index by λ_1.

Once we have placed the $j-1$st k into the sequence in such a way that it increases the major index by λ_{j-1}, we placed the next k as follows. If $\lambda_j - j \geq 0$, then we find the first non-descent to the left of the last placed k where inserting a k increases the total major index by exactly λ_j. Note that such a position always exists because placing the k in the position to the left of the entry immediately to the left of the last k will increase the major index by exactly λ_{j-1} if it has not been placed inside a pre-existing descent. If this would put it inside a pre-existing descent, then putting this new k in the first position to left of that position that is not a pre-existing descent will increase the major index by exactly λ_{j-1}. We now keep moving to the left until the amount that we increase the major index is exactly λ_j.

If $\lambda_j > j$, then we place the jth k inside the λ_jth descent as counted from the right. This increases the major index by exactly λ_j.

To reverse this bijection, we read the word from right to left, removing those ks that sit inside descents and keeping track of the amount by which the major index decreases with each removal. Then we read the word from left to right, removing all remaining ks and keeping track of the amount by which the major index decreases.

3.2.5 One proof is to assign to each permutation σ the companion, σ', obtained by interchaning the images of 1 and 2: $\mathcal{I}(\sigma') = \mathcal{I}(\sigma) \pm 1$. Another is to set $q = -1$ in equation (3.20).

3.2.6 The inversion number of a permutation of n letters is equal to the number of positions to the right of n (which equals the number of transpositions required to move n to the last position on the right) plus the inversion number of the word in $n-1$ letters obtained by removing that n.

3.2.7 Choose any two of the n positions in the sequence. there are $\binom{n}{2}$ such choices. In exactly half of the permutations, $n!/2$ permutations, these two positions contribute 1 to the inversion number.

3.2.9 If $k \in r_{i+1} - r_i$, then there is a 1 in row $i+1$, column k. The set $[[k-1]] - r_{i+1}$ is the subset of $\{1, \ldots, k-1\}$ for which these columns have more 1s than -1s in the rows below $i+1$. If $k \in r_i - r_{i+1}$, then there is a -1 in row $i+1$, column k.

3.2.11 Nothing is known about these polynomials. They appear to be monotonic.

3.2.12 See page 193 for what is known about these polynomials in k, and exercises 6.1.3 and 6.1.4 for some conjectures.

3.2.14 $(1+q)^{n(n-1)/2}$ (see exercise 3.5.9).

3.2.15 For each product of entries that is -1, there are two uniquely defined products that are $+1$: If there is a $+1$ to the northeast of a -1, then there is a $+1$ directly south of that -1 and another $+1$ directly west; if there is a -1 northeast of a $+1$, then there is another $+1$ directly north of that -1 and another $+1$ directly east.

3.3.1

$$\prod_{(i,j,k)\in\mathcal{B}(r,s,t)} \frac{1-q^{i+j+k-2}}{1-q^{i+j+k-1}} = \prod_{\substack{1\le i\le r\\ 1\le j\le s}} \frac{1-q^{i+j+t-1}}{1-q^{i+j-1}} \longrightarrow \prod_{i,j=1}^{\infty} \frac{1}{1-q^{i+j-1}}.$$

There are are exactly k representations of $k+1$ as a sum of two positive integers (counting order).

3.3.3 Let σ' agree with σ except that $sigma'(i) = \sigma(i+1)$ abd $sigma'(i+1) = \sigma(i)$. then we have

$$f(\sigma) - f(\sigma') = i(i - sigma(i)) + (i+1)(i+1-\sigma(i+1)) - i(i - sigma(i+1)) - (i+1)(i+1-\sigma(i)) = \sigma(i) - \sigma(i+1).$$

Therefore, f is such a function. Since every permutation can be built from adjacent transpositions, such a function is unique.

3.4.5 Let $A = (a_{ij})_{i,j=1}^n$ and $a_{nj} = \sum_{k=1}^t a_{nj}^{(k)}$. Then

$$\begin{aligned} \det A &= \sum_{\sigma \in \mathcal{S}_n} (-1)^{\mathcal{I}(\sigma)} \prod_{i=1}^n a_{i\,\sigma(i)} \\ &= \sum_{\sigma \in \mathcal{S}_n} (-1)^{\mathcal{I}(\sigma)} \left(\prod_{i=1}^{n-1} a_{i\,\sigma(i)} \right) \left(\sum_{k=1}^t a_{n\,\sigma(n)}^{(k)} \right) \\ &= \sum_{k=1}^t \sum_{\sigma \in \mathcal{S}_n} (-1)^{\mathcal{I}(\sigma)} \prod_{i=1}^n a_{i\,\sigma(i)}^{(k)} \\ &= \sum_{k=1}^t \det A_k. \end{aligned}$$

3.4.6 Let $j-k$ be the y-co-ordinate of the first point on the path that touches the line $x = r-1$.

3.4.7 Same proof as exercise 3.4.6, but instead of just counting the number of lattice paths, keep track of the partition that each lattice path encodes.

3.5.1

$$\begin{pmatrix} 4 & -2 & -5 \\ 1 & 3 & 5 \\ 2 & -1 & 11 \end{pmatrix} \longrightarrow \begin{pmatrix} 7 & 5 \\ 7 & 38 \end{pmatrix} \longrightarrow (77).$$

3.5.3 the proof exactly follows that of theorem 3.12 except that $M^\star$ is constructed by replacing the last $n-m$ columns of M^C by the last $n-m$ columns of the identity matrix.

3.5.6

$$\left| x_j^{n-i} \right|_\lambda = \Bigg[\prod_{1<i<j\le n} (x_i + \lambda x_j) \cdot x_1 \cdots x_{n-1} \cdot \prod_{1\le i<j<n} (x_i + \lambda x_j)$$

$$
\begin{aligned}
&\quad + \lambda \prod_{1<i<j\le n} (x_i + \lambda x_j) \cdot x_2 \cdots x_n \cdot \prod_{1\le i<j<n} (x_i + \lambda x_j) \Bigg] \\
&\quad \div \left[x_2 \cdots x_{n-1} \cdot \prod_{1<i<j<n} (x_i + \lambda x_j) \right] \\
&= \prod_{1\le i<j<n} (x_i + \lambda x_j) \prod_{1<i<n} (x_i + \lambda x_n)\ (x_1 + \lambda x_n).
\end{aligned}
$$

3.5.7 If $B_{ij} = 1$ and there are no -1s directly below it, then j appears $n-i+1$ times in the monotone triangle. If $B_{ij} = 1$ and $B_{kj} = -1$ with $B_{rj} = 0$ for $i < r < k$, then j appears $k - i$ times in the monotone triangle and then disappears as you read down successive rows. Therefore $\sum_i (n - i)B_{ij}$ is one less than the number of times j appears.

4.1.1 $7+6+6+5+1+1+1+1+1$ precedes and $7+6+6+4+4+1+1$ follows.

4.1.2 Using the generating functions, e_n is the coefficient of t^n in $(1+t)^m$, which is $\binom{m}{n}$, and h_n is the coefficient of t^n in $(1-t)^{-m}$, which is $\binom{m+n-1}{n}$.

4.1.3 This follows from the fact that every permutation is a composition of adjacent transpositions and an adjacent transposition changes the parity of the inversion number.

4.1.5 Invert the matrix

$$
\begin{pmatrix}
1 & 4 & 6 & 12 & 24 \\
0 & 1 & 2 & 5 & 12 \\
0 & 0 & 1 & 2 & 6 \\
0 & 0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$

4.1.6 Use the generating function and the result of exercise 3.1.19 on page 83.

4.1.7 Use the generating function and Theorem 3.3 on page 78.

4.1.8 Equation (4.6) is Proposition 4.2 with λ equal to k 1s. To prove equation (4.7), expand on the first row and use induction:

$$\det(e_{1-i+j}) = e_1 h_{n-1} - e_2 h_{n-2} + \cdots + (-1)^{n-1} e_n h_0.$$

It follows from the generating functions that for $n \ge 1$:

$$e_0 h_n - e_1 h_{n-1} + e_2 h_{n-2} - \cdots + (-1)^n e_n h_0 = 0.$$

4.1.9 Note correction: Last term in the displayed summation should be $(-1)^b h_{a+b+1}$. Use equation (4.4) and expand on the first row.

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4.2.5 We encode each column of the semistandard tableau as a lattice path. Lattice paths are now permitted to touch at vertices, but may not share edges. the proof proceeds exactly as in the proof of Theorem 4.3, except that we switch the tails of two paths that share a common edge.

4.2.6 Let $x_i = q^{n+1-i}$ so that an x_i in a given square of the Young diagram represents a stack of $n+1-i$ cubes above that position. then take the limit as $n \to \infty$.

4.2.7 Let $x_i = q^{2n+1-2i}$.