Hints, Comments, and Solutions for Exercises in

Proofs and Confirmations

July 13, 1999

1.1.5 This turned out to be much more difficult than I anticipated. It is easy to see that any subset that contains all permutation matrices cannot contain any ASM's with any $-1$'s (otherwise, we can permute the columns so that we get a $-1$ in the first column). I believe that given any ASM with at least one $-1$, some power of it must fail to be an ASM. Heuristically, taking powers increases the number of $-1$'s and/or moves the $-1$'s closer to the boundary, but I would very much appreciate a simple proof that any ASM with a $-1$ in it has a power that fails to be an ASM.

1.1.12 The conjecture is that $A_{n,k}$ can be odd if and only if $n = (4^t - 1)/3, (4^t + 2)/3, (2 \cdot 4^t + 1)/3$ or $(2 \cdot 4^t + 4)/3$ for some integer $t \geq 1$. If $n = (4^t - 1)/3$, then $A_{n,k}$ is odd if and only if $k = (n - 1)/2, (n + 1)/2, or (n + 3)/2$. If $n = (2 \cdot 4^t + 1)/3$, then $A_{n,k}$ is odd if and only if $k = (n + 1)/2$. Let $A_{\{t\}}$ (respectively, $B_{\{t\}}$) be the set of values of $k$ for which $A_{n,k}$ is odd when $n = (4^t + 2)/3$ (respectively, $n = 2 \cdot (4^t + 2)/3$). Then $A_{\{t\}} = \{k | k \in A_{\{t-1\}}$ or $n+1-k \in A_{\{t-1\}}\}$ (respectively, $B_{\{t\}} = \{k | k \in B_{\{t-1\}}$ or $n+1-k \in B_{\{t-1\}}\}$).
1.1.13 I do not have a proof for this.

1.1.14 I do not know of a slick proof. It can be forced through by considering the three cases that depend on the residue class of $n$ modulo 3, simplifying the rational product, and then demonstrating that an arbitrary prime power divides the numerator at least as often as it divides the denominator.

1.2.2 All terms on the right cancel except those with $k = 1$ in the denominator and with $k = t$ in the numerator.

1.2.3 $1085787; 90379784; 267227532$

1.2.5 $r$ orbits of size 1, no orbits of size 2, $r(r - 1)$ orbits of size 3, no orbits of size 6.

1.2.6 $r$ orbits of size 1, no orbits of size 2, $(r^3 - r)/3$ of size 3, no orbits of size 6.

1.2.9 $1185; 151008$

1.2.10 The product in question is equal to

$$\prod_{d=1}^\infty \frac{1 - q^{2d}}{1 - q^{2d-1}} \times \prod_{d=1}^\infty \frac{1 - q^{2d}}{1 - q^{2d-1}} = (1 + q^2)(1 + q^6)(1 + q^9)(1 + q^{15})(1 + q^{18}) + q^{14}(1 - q^4 + q^8)(1 + q^5 + q^{10})^2$$

1.2.11 The orbits of $B(r,r,r)/S_3$ that are not in $B(r-1,r-1,r-1)/S_3$ correspond to lattice points $(r,i,j)$ with $1 \leq i \leq j \leq r$:

$$TSPP(r) - TSPP(r-1) = r \prod_{i=1}^r \prod_{j=i}^{r-1} (i + j + r - 1)$$

1.2.12 $\emptyset, 1, 2 1, 2 2$.
1.3.2 (a) \[ 3 \times 3 \times 2 \]

1.3.3 (a) \[ 30; (b) 19 + 3(6+5+4+3+3+2) + 16 + 3(3+3+2+1+1) + 7 + 3(2+1) + 4 = 154 \]

1.3.4 \[ \sum_{j} (3a_{j,j} - 2 + 3(a_{j,j} + 1) + \cdots + a_{j,r})^2) \]

1.3.5 Hint: Use the fact that \[ 1 - q^{a_{j,j}} = (1 - q^{j}) + q^{j}(1 - q^{a_{j,j} - j}). \]

1.3.9 \[ \prod_{1 \leq i \leq j \leq r} (r + 2i + j - 1)! \]

1.3.11 This is nonsense. Every descending plane partition satisfies the definition of being stripped.

2.1.2 5; 7

2.1.3 It is equal to the number of partitions of \( n \) into at most two parts. The larger of these two parts can be any integer that is greater than or equal to \( n/2 \), and there are \( 1 + \lfloor n/2 \rfloor \) possibilities. 

2.1.4 This is the number of partitions of \( n \) into at most three parts which is the integer closest to \( (n + 3)/2 \). We have that

\[ \frac{q(b - 1) + (q^2b - 1) + (q^3b - 1) + \cdots + (q^{r}b - 1)}{q} = \frac{(q^r - 1)(q^r - 1)(q^r - 1)}{1} \]

Comparing coefficients of \( q^n \), we see that the desired number is equal to

\[ \frac{1}{q} \left( \frac{q - 1}{q - 1} + \frac{q^2 - 1}{q^2 - 1} + \frac{q^3 - 1}{q^3 - 1} \right) \]

The general problem of the number of partitions of \( n \) into at most \( k \) parts is discussed in example 2, chapter 5 and example 1, chapter 12 of Andrews, The Theory of Partitions.
2.1.10 First assume both signs are the same, then we can write the equality as
\[ 3(j^2 - k^2) = \mp (j - k). \]

If \( j \neq k \), then we can divide both sides by \( j - k \) to get that \( 3(j + k) = \pm 1 \). Again, this cannot be.

2.2.9 If there is a 0 in the second partition, then interchange the two partitions.

2.2.10 This is the Jacobi triple product identity with \( q^4 \) replaced by \( q \) and \( x = -q^{-1} \).
\[ (t) \rho_{\sigma} + \left( (t) \rho_{\sigma} - \sum_{a \in \sigma}^{0} \right) = \]

\[ \Psi_{x \rho + 1} (1 - x) \sum_{a \in \sigma}^{0} \psi_{a} = (f) \psi_{\sigma} \]

On the other hand,
\[
(\alpha \cdot (f)) = (f) \rho_{\sigma} \text{ \( \text{where } \alpha \) is odd, then } \]

\[ (u) = (u) \rho_{\sigma} \text{ \( \text{where } f \) is even. then } \]

The proof is then by induction, using the fact that the first equality and then the second:
\[
(\alpha \cdot (f)) = (f) \rho_{\sigma} \]

\[ (\alpha \cdot (f)) = (u) \rho_{\sigma} \]

\[ (\alpha \cdot (f)) = (u) \rho_{\sigma} \]

\[
2.2.13 \text{ Using the first equality and then the second:} \]
\[
\sum_{m=1}^{\infty} \sum_{m=1}^{\infty} (1 + q^k) (1 - q^2k - 1) = (f) \psi_{\sigma} \]

\[ 2.2.15 \text{ Equation (2.13) simplifies to} \]
\[ \sum_{k=1}^{\infty} \equiv \sum_{k=1}^{\infty} (1 + q^k) (1 - q^2k - 1) = (f) \psi_{\sigma} \]

\[ 2.3.2 \text{ No.} \]
\[ 2.3.4 \text{ Taking the derivative of the generating function, we get} \]
\[ \sum_{m=1}^{\infty} \sum_{m=1}^{\infty} (1 + q^k) (1 - q^2k - 1) = (f) \psi_{\sigma} \]

\[ 2.3.6 \text{ Exercises 2.3.4 and 2.3.5 imply that} \]
\[ \sum_{k=1}^{\infty} \sum_{k=1}^{\infty} (1 + q^k) (1 - q^2k - 1) = (f) \psi_{\sigma} \]

\[ 2.3.6 \text{ Taking the derivative of the generating function, we get} \]
\[ \sum_{k=1}^{\infty} \sum_{k=1}^{\infty} (1 + q^k) (1 - q^2k - 1) = (f) \psi_{\sigma} \]

\[ 2.3.6 \text{ Equation (2.13) simplifies to} \]
\[ \sum_{k=1}^{\infty} \sum_{k=1}^{\infty} (1 + q^k) (1 - q^2k - 1) = (f) \psi_{\sigma} \]

\[ 2.3.6 \text{ Using the first equality and then the second:} \]
\[ \sum_{m=1}^{\infty} \sum_{m=1}^{\infty} (1 + q^k) (1 - q^2k - 1) = (f) \psi_{\sigma} \]

\[ 2.3.6 \text{ Taking the derivative of the generating function, we get} \]
\[ \sum_{k=1}^{\infty} \sum_{k=1}^{\infty} (1 + q^k) (1 - q^2k - 1) = (f) \psi_{\sigma} \]

\[ 2.3.6 \text{ Equation (2.13) simplifies to} \]
\[ \sum_{k=1}^{\infty} \sum_{k=1}^{\infty} (1 + q^k) (1 - q^2k - 1) = (f) \psi_{\sigma} \]

\[ 2.3.6 \text{ Taking the derivative of the generating function, we get} \]
\[ \sum_{k=1}^{\infty} \sum_{k=1}^{\infty} (1 + q^k) (1 - q^2k - 1) = (f) \psi_{\sigma} \]
2.3.7 \[ \sum_{n=1}^{\infty} a_k(n)x^n = \left( \frac{f}{f+x} \right)^n \] 

The determinant on the right is an alternating polynomial in the $x^j$ of degree $n$, and so it is equal to the Vandermonde product difference $z/(1-z)$.

\[ \det \left( \left( \frac{f+x}{f} \right)^n \right) = \left( \frac{f}{f+x} \right)^n \]

2.3.14 (a) 
\[
\begin{pmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & -1 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

2.3.15 It is the sum over all rows, except the bottom row, of the number of integers in that row that do not appear in the row below.

2.3.18 For given positive integers $a_1, \ldots, a_j$, which is the coefficient of $q^0$ in the product $\prod_{i=1}^{j-1} (1-q^i)^{a_i}$?

2.4.7 \[ \det ( (x_i+j)(x_i+j) \cdots (x_i+j) ) = \det ( (x_i+i+1)(x_i+i+1) \cdots (x_i+i+j) ) \]

The determinant on the right is an alternating polynomial in the $x^j_i$ of total degree $n(n-1)/2$ and so it is equal to the Vandermonde product times a constant which must be $(1-\epsilon)^n$. We have that

\[ \det ( (x_i+j)(x_i+j) \cdots (x_i+j) ) = (1-\epsilon)^n \]

2.3.19 Given the power series $1 + \sum_{r=1}^{\infty} b_r q^r$, let $a_1 = b_1$. The coefficients of $q^0$ and $q^1$ agree in the power series with those in the expansion of $(1-q)^{-b_1}$. Assume that we have found unique integers $a_1, \ldots, a_{j-1}$ such that the coefficients of $q^0, q^1, \ldots, q^{j-1}$ in the product

\[ \prod_{i=1}^{j-1} (1-q^i)^{a_i} \]

agree with those in the power series, and let $c_j$ be the coefficient of $q^j$ in the expansion of this product. We have that

\[ c_j = b_j - c_j \]

where $\ell \geq 0$, $j \leq 6$, and $\ell$ does not divide $j$. We have that $a_1 = \ell q$ where $q < u$ and for $u \geq 0 = (u)^n \cdot a_1 - (u)^n q $

\[ \left( \frac{y^2-1}{y^2} - x - 1 \right) \left( \frac{y^2-u^2}{y^2} \right) q = \frac{y}{u} x(u)^n \cdot a_1 - (u)^n \cdot a_1 \]
3.1.3 Using the recursive formula, we have that
\[ f(q, m) = 1 - \sum_{\ell=1}^{m} (m - 1)^{\ell} - m^{m - 1} \sum_{\ell=1}^{m} (m - 1)^{\ell} + (m^{m - 1} - 1) \]
\[ = m \sum_{j=1}^{\infty} (1 - q^{m - j})^{m - j} - \cdots \]
\[ = (1 - q^{m - 1}) f(q, m - 1) . \]

3.1.4 Note that \( \alpha_k = 1. \)

3.1.5 \( \alpha_j^{2 + j} = \alpha_j^{2 + j} (k + 1) \) and the sum in \( G(\alpha) \) can be taken over any set of residues modulo \( k. \)

3.1.6 \( \alpha[k + 1] / 2 = \alpha(k - 1) / 4 = \alpha + 3 + \cdots + k - 2. \)

3.1.10 Note that the conditions should read "positive for \( k \equiv \pm 1 \pmod{8} \) and negative for \( k \equiv \pm 3 \pmod{8}. " \) The sign of the product is positive if there are an even number of integers \( j \) in the interval \( (k/4, (k - 1)/2) \) and negative if there are an odd number of integers in this interval.

3.1.13 For \( j = 2, \) this identity is
\[ (n + 2 / 2) = q^{2 / 2} [n / 2] q^{n / 2} + [n / 2] q^{n - 1 / 1} . \]

3.1.19 The coefficient of \( x_i \) is the generating function for partitions into exactly \( i \) parts, all of which are \( \leq n. \)

3.1.20
\[ i \prod_{j=1}^{\infty} \left( 1 - q^{n - i + j} \right) = \prod_{j=1}^{i} (1 - q^{n - i + j}) . \]
It follows that

\[
\left[ n \right]_q \frac{(i - 1)}{2} \left( -x \right) - i \left( q^n - i + 1 ; q \right)_i = \left( q - n ; q \right)_i \left( q ; q \right)_i \frac{(i - 1)}{2} \left( x q^n \right)
\]

Note that if \( i > n \), then one of the factors in \( \left( q - n ; q \right)_i \) is 1.

3.1.21 \( \lim_{n \to \infty} \left[ n \right]_q = \lim_{n \to \infty} \left( q^n - i + 1 ; q \right)_i \left( q ; q \right)_i \frac{(i - 1)}{2} \left( -x \right) = 1 \left( q ; q \right)_i \)\)

3.2.1 \( \text{maj}(11001010100) = 2 + 5 + 7 + 9 = 23. \)

\( I(01101011001) = 2 + 3 + 5 + 5 = 15. \)

\( I(11001010100) = 2 + 2 + 3 + 4 + 5 + 5 = 21. \)

3.2.4 We proceed by induction on the number of variables. The previous exercise established this result for \( k = 2. \) Assume that it is correct up to \( k - 1, k \geq 3. \) Let \( \sigma \) be a sequence with \( \max_i \leq k \), and let \( \lambda \) be a partition into at most \( \max_i \leq \sum \lambda_i \leq \sum \lambda_i \leq \sum \lambda_i \leq \sum \lambda_i \) parts less than or equal to \( M = \sum \lambda_i \).

We must establish a one-to-one correspondence between the set of pairs \( \left( \sigma, \lambda \right) \) and the set of sequences, \( \tau \), with \( \max_i \leq k \leq \sum \tau_i \leq \sum \tau_i \) such that \( \text{maj}(\sigma) + |\lambda| = \text{maj}(\tau). \) It then follows that

\[
\sum_{\tau} q^{\text{maj}(\tau)} = \sum_{\sigma} q^{\text{maj}(\sigma)} \sum_{\lambda} q^{|\lambda|} = \left[ 1 + \cdots + 1 \right] \left[ q + \cdots + q \right] = q + \cdots + q + q + \cdots + q = \left( \text{cond} b \right) + \cdots + \left( \text{cond} b \right)
\]

\[
\sum_{\sigma} q^{\text{maj}(\sigma)} \sum_{\lambda} q^{|\lambda|} = \left[ 1 + \cdots + 1 \right] \left[ q + \cdots + q \right] = q + \cdots + q + q + \cdots + q = \left( \text{cond} b \right) + \cdots + \left( \text{cond} b \right)
\]

We begin with the largest part in \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m = 0). \) If the entry in \( \sigma \) in position \( \lambda_1 - 1 \) is less than or equal to the next entry, then insert \( k \) after the entry in position \( \lambda_1 - 1. \) Otherwise, insert \( k \) in the first position to the left where it does not fall inside a pre-existing descent.

Note that wherever we put this first \( k \), we have increased the major index by \( \lambda_1. \)

Once we have placed the \( j - 1 \)th \( k \) into the sequence in such a way that

\[
\sum_{\tau} q^{\text{maj}(\tau)} = \sum_{\sigma} q^{\text{maj}(\sigma)} \sum_{\lambda} q^{|\lambda|} = \left[ 1 + \cdots + 1 \right] \left[ q + \cdots + q \right] = q + \cdots + q + q + \cdots + q = \left( \text{cond} b \right) + \cdots + \left( \text{cond} b \right)
\]

We proceed by induction on the number of variables. The previous exercise established this result for \( k = 2. \) Assume that it is correct up to \( k - 1, k \geq 3. \) Let \( \sigma \) be a sequence with \( \max_i \leq k \), and let \( \lambda \) be a partition into at most \( \max_i \leq \sum \lambda_i \leq \sum \lambda_i \) parts less than or equal to \( M = \sum \lambda_i \).

We must establish a one-to-one correspondence between the set of pairs \( \left( \sigma, \lambda \right) \) and the set of sequences, \( \tau \), with \( \max_i \leq k \leq \sum \tau_i \leq \sum \tau_i \) such that \( \text{maj}(\sigma) + |\lambda| = \text{maj}(\tau). \) It then follows that

\[
\sum_{\tau} q^{\text{maj}(\tau)} = \sum_{\sigma} q^{\text{maj}(\sigma)} \sum_{\lambda} q^{|\lambda|} = \left[ 1 + \cdots + 1 \right] \left[ q + \cdots + q \right] = q + \cdots + q + q + \cdots + q = \left( \text{cond} b \right) + \cdots + \left( \text{cond} b \right)
\]

We begin with the largest part in \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m = 0). \) If the entry in \( \sigma \) in position \( \lambda_1 - 1 \) is less than or equal to the next entry, then insert \( k \) after the entry in position \( \lambda_1 - 1. \) Otherwise, insert \( k \) in the first position to the left where it does not fall inside a pre-existing descent.

Note that wherever we put this first \( k \), we have increased the major index by \( \lambda_1. \)

Once we have placed the \( j - 1 \)th \( k \) into the sequence in such a way that

\[
\sum_{\tau} q^{\text{maj}(\tau)} = \sum_{\sigma} q^{\text{maj}(\sigma)} \sum_{\lambda} q^{|\lambda|} = \left[ 1 + \cdots + 1 \right] \left[ q + \cdots + q \right] = q + \cdots + q + q + \cdots + q = \left( \text{cond} b \right) + \cdots + \left( \text{cond} b \right)
\]

We proceed by induction on the number of variables. The previous exercise established this result for \( k = 2. \) Assume that it is correct up to \( k - 1, k \geq 3. \) Let \( \sigma \) be a sequence with \( \max_i \leq k \), and let \( \lambda \) be a partition into at most \( \max_i \leq \sum \lambda_i \leq \sum \lambda_i \) parts less than or equal to \( M = \sum \lambda_i \).

We must establish a one-to-one correspondence between the set of pairs \( \left( \sigma, \lambda \right) \) and the set of sequences, \( \tau \), with \( \max_i \leq k \leq \sum \tau_i \leq \sum \tau_i \) such that \( \text{maj}(\sigma) + |\lambda| = \text{maj}(\tau). \) It then follows that

\[
\sum_{\tau} q^{\text{maj}(\tau)} = \sum_{\sigma} q^{\text{maj}(\sigma)} \sum_{\lambda} q^{|\lambda|} = \left[ 1 + \cdots + 1 \right] \left[ q + \cdots + q \right] = q + \cdots + q + q + \cdots + q = \left( \text{cond} b \right) + \cdots + \left( \text{cond} b \right)
\]

We begin with the largest part in \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m = 0). \) If the entry in \( \sigma \) in position \( \lambda_1 - 1 \) is less than or equal to the next entry, then insert \( k \) after the entry in position \( \lambda_1 - 1. \) Otherwise, insert \( k \) in the first position to the left where it does not fall inside a pre-existing descent.

Note that wherever we put this first \( k \), we have increased the major index by \( \lambda_1. \)
now keep moving to the left until the amount that we increase the major index +1 directly east.

3.2.5 One proof is to assign to each permutation \( \sigma \) the companion, \( \sigma' \), obtained by interchanging the images of 1 and 2:

\[
I(\sigma') = I(\sigma) \pm 1.
\]

Another is to set \( q = -1 \) in equation (3.20).

3.2.6 The inversion number of a permutation of \( n \) letters is equal to the number of positions to the right of \( n \)(which equals the number of transpositions required to move \( n \) to the last position on the right) plus the inversion number of the word in \( n - 1 \) letters obtained by removing that \( n \).

3.2.7 Choose any two of the \( n \) positions in the sequence. there are \( \binom{n}{2} \) such choices. In exactly half of the permutations, \( n! / 2 \) permutations, these two positions contribute 1 to the inversion number.

3.2.9 If \( k \in r_{i+1} - r_i \), then there is a 1 in row \( i+1 \), column \( k \). The set \( \{k-1\} - r_{i+1} \) is the subset of \( \{1, \ldots, k-1\} \) for which these columns have more 1s than \(-1\)s in the rows below \( i+1 \). If \( k \in r_i - r_{i+1} \), then there is a \(-1\) in row \( i+1 \), column \( k \).

3.2.11 Nothing is known about these polynomials. They appear to be monotonic.

3.2.12 See page 193 for what is known about these polynomials in \( k \), and exercises 6.1.3 and 6.1.4 for some conjectures.

3.2.14 (1 + \( q \)\(n\))\((n-1)/2\) (see exercise 3.2).

3.2.15 For each product of entries that is \(-1\), there are two mutually defined

\[
(\text{see exercise 3.2}).
\]

3.2.16 For some coefficients,\n
3.2.17 Some unknown about these polynomials. They appear to be mono-

3.2.18 See page 193 for what is known about these polynomials in \( k \), and

3.2.19 Another is to set \( r \) in equation (3.20).}

\[
(\rho(z),q) = (\rho(z),q) \rightarrow \infty
\]

\[
\prod_{l \leq i \leq r} (1 - q_i + j - k - 1)
\]

\[
\prod_{l \geq i \leq r} (1 - q_i + j - k - 1)
\]

\[
\prod_{l = 1}^{q_{i+1} - q_i} (1 - q_i + j - k - 1)
\]

\[
\prod_{l = 1}^{q_{i+1} - q_i} (1 - q_i + j - k - 1)
\]

3.2.17 Choose any two of the \( n \) positions in the sequence. These are such \( \binom{n}{2} \) such choices. In exactly half of the permutations, \( n! / 2 \) permutations, these two positions contribute 1 to the inversion number.

3.2.18 See page 193 for what is known about these polynomials in \( k \), and exercises 6.1.3 and 6.1.4 for some conjectures.
There are exactly \( k \) representations of \( k + 1 \) as a sum of two positive integers (counting order).

3.3.3 Let \( \sigma' \) agree with \( \sigma \) except that \( \sigma'(i) = \sigma(i + 1) \) and \( \sigma'(i + 1) = \sigma(i) \). Then we have

\[
\frac{f(\sigma)}{f(\sigma')} = \frac{i(\sigma) - \sigma(i) + (i + 1)(\sigma(i + 1) - \sigma(i))}{(i + 1)(\sigma(i) + \sigma(i + 1)) - (i + 1)(\sigma(i) - \sigma(i + 1))} = \frac{\sigma(i) - \sigma(i + 1)}{\sigma(i) + \sigma(i + 1)}.
\]

Therefore, \( f \) is such a function. Since every permutation can be built from adjacent transpositions, such a function is unique.

3.4.5 Let \( A = (a_{ij})_{n \times n} \) and \( a_{nj} = \sum_{k=1}^{t} a_{(k)}nj \). Then

\[
\det A = \sum_{\sigma \in S_n} (-1)^{\ell} \prod_{i=1}^{n} a_{i\sigma(i)} = \sum_{\sigma \in S_{n-1}} (-1)^{\ell} \prod_{i=1}^{n-1} a_{i\sigma(i)} \sum_{k=1}^{t} a_{(k)n\sigma(n)}.
\]

3.4.6 Let \( j-k \) be the \( y \)-coordinate of the first point on the path that touches the line \( x = r-1 \).

3.4.7 Same proof as exercise 3.4.6, but instead of just counting the number of lattice paths, keep track of the partition that each lattice path encodes.

3.5.1

\[
\begin{bmatrix} 4 & -2 & -5 \\ 1 & 3 & 5 \\ 2 & -1 & 11 \end{bmatrix} \rightarrow (7, 5) \rightarrow (77).
\]

3.5.3 The proof exactly follows that of theorem 3.12 except that \( M \odot \) is constructed by replacing the last \( n-m \) columns of \( M \) with the last \( n-m \) columns of identity matrix.

3.5.6

\[
\begin{vmatrix} x_n - i \end{vmatrix} = \left( \prod_{1 \leq i < j \leq n} (x_i + \lambda x_j) \cdot x_1 \cdot \ldots \cdot x_{n-1} \cdot \prod_{1 \leq i < j < n} (x_i + \lambda x_j) \right)^{\frac{-1}{\lambda}}.
\]

Therefore, \( f \) is such a function. Since every permutation can be built from adjacent transpositions, such a function is unique.

3.3.3 Let \( \sigma' \) agree with \( \sigma \) except that \( \sigma'(i) = \sigma(i + 1) \) and \( \sigma'(i + 1) = \sigma(i) \). Then we have

\[
\frac{f(\sigma)}{f(\sigma')} = \frac{i(\sigma) - \sigma(i) + (i + 1)(\sigma(i + 1) - \sigma(i))}{(i + 1)(\sigma(i) + \sigma(i + 1)) - (i + 1)(\sigma(i) - \sigma(i + 1))} = \frac{\sigma(i) - \sigma(i + 1)}{\sigma(i) + \sigma(i + 1)}.
\]

Therefore, \( f \) is such a function. Since every permutation can be built from adjacent transpositions, such a function is unique.
1

4.1.9 Note correction: Last term in the displayed summation should be

\[
\lambda = 0 = \lambda \eta - (I - \eta) + \cdots - \zeta = \eta - (I - \eta) + \cdots - \zeta = \eta - (I - \eta) + \cdots - \zeta
\]

4.1.8 Equation (4.7) is Proposition 4.2, with A equal to 1. To prove equation

4.1.7 Use the generating function and Theorem 3.3 on page 78.

4.1.6 Use the generating function and the result of exercise 3.1.19 on page 83.

4.1.5 Invert the matrix

\[
\begin{pmatrix}
1 & 4 & 6 & 12 & 24 \\
0 & 1 & 2 & 5 & 12 \\
0 & 0 & 1 & 2 & 6 \\
0 & 0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

4.1.4 Use the generating function and the result of exercise 3.1.19 on page 83.

4.1.3 This follows from the fact that every permutation is a composition of adjacent transpositions and an adjacent transposition changes the parity of the inversion number.

4.1.2 Using the generating functions, \( e_n \) is the coefficient of \( t^n \) in \( (1 + t)^m \), which is \( \binom{m}{n} \), and \( h_n \) is the coefficient of \( t^n \) in \( (1 - t)^{-m} \), which is \( \binom{m + n - 1}{n} \).

4.1.1 Use equation (4.4) and expand on the first row.

4.1.0 Note correction: Last term in the displayed summation should be

\[
\lambda = 0 = \lambda \eta - (I - \eta) + \cdots - \zeta = \eta - (I - \eta) + \cdots - \zeta
\]

4.1.0 This follows from the fact that every permutation is a composition of adjacent transpositions and an adjacent transposition changes the parity of the inversion number.

4.0.9 Note correction: Last term in the displayed summation should be

\[
\lambda = 0 = \lambda \eta - (I - \eta) + \cdots - \zeta = \eta - (I - \eta) + \cdots - \zeta
\]

4.0.8 Equation (4.7) is Proposition 4.2, with A equal to 1. To prove equation

4.0.7 Use the generating function and Theorem 3.3 on page 78.

4.0.6 Use the generating function and the result of exercise 3.1.19 on page 83.

4.0.5 Invert the matrix

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
4 & 1 & 0 & 0 & 0 \\
6 & 2 & 1 & 0 & 0 \\
12 & 2 & 1 & 0 & 0 \\
24 & 1 & 0 & 0 & 0
\end{pmatrix}
\]

4.0.4 Use the generating function and the result of exercise 3.1.19 on page 83.

4.0.3 This follows from the fact that every permutation is a composition of adjacent transpositions and an adjacent transposition changes the parity of the inversion number.

4.0.2 Using the generating functions, \( e_n \) is the coefficient of \( t^n \) in \( (1 + t)^m \), which is \( \binom{m}{n} \), and \( h_n \) is the coefficient of \( t^n \) in \( (1 - t)^{-m} \), which is \( \binom{m + n - 1}{n} \).

4.0.1 Use equation (4.4) and expand on the first row.

4.0.0 Note correction: Last term in the displayed summation should be

\[
\lambda = 0 = \lambda \eta - (I - \eta) + \cdots - \zeta = \eta - (I - \eta) + \cdots - \zeta
\]
4.2.2 We encode each column of the semistandard tableau as a lattice path. Lattice paths are now permitted to touch at vertices, but may not share edges. The proof proceeds exactly as in the proof of Theorem 4.3, except that we switch the tails of two paths that share a common edge.

4.2.5 Let $x_i = q^n + 1 - i$ so that an $x_i$ in a given square of the Young diagram represents a stack of $n + 1 - i$ cubes above that position. Then take the limit as $n \to \infty$.

4.2.6 Let $x_i = q^{2n + 1 - 2i}$. Then:

$$t_2 \left( \sum_{i=1}^{\infty} x_i \right) = x_1 + x_2 + \cdots$$