Math 377, Handout 3:
Problems due September 29

1. Let \( f(x) = x^2 \sin(1/x), \) \( F(x) = x. \) Each of these functions approaches 0 as \( x \) approaches 0, but if we try to apply l'Hospital's rule we get that the limit as \( x \) approaches 0 of \( f'(x)/F'(x) \) is undefined, while it is clear that
\[
\lim_{x \to 0} \frac{x^2 \sin(1/x)}{x} = \lim_{x \to 0} x \sin(1/x) = 0.
\]
What is wrong with this example?

2. Prove that if \( f \) is continuous on a closed interval \([a, b]\), differentiable on the open interval \((a, b)\), and if \( f(a) = f(b) = 0 \), then for any real number \( \alpha \) there is an \( x \in (a, b) \) such that
\[
\alpha f(x) + f'(x) = 0.
\]
Hint: Apply the Mean Value Theorem to the function \( g(x) = e^{\alpha x} f(x) \).

3. Let \( P(x) \) be any polynomial of degree at least two, all of whose roots are real. Prove that all of the roots of \( P'(x) \) must be real. Hint: Let \( r_1 \leq r_2 \leq \ldots \leq r_n \) be the roots of \( P(x) \). Note that some of these roots may be equal (if the polynomial has any roots of multiplicity > 1). How many roots does \( P'(x) \) have (counting multiple roots)? Show that if \( r_i < r_{i+1} \), then \( P' \) has a real root between \( r_i \) and \( r_{i+1} \). Show that if \( P \) has a multiple root of multiplicity \( t \geq 2 \) at \( x = a \), then \( P' \) must have a root at \( x = a \). What is the multiplicity of the root of \( P' \) at \( x = a \)? Now use a counting argument to show that all the roots of \( P' \) must be real.

4. Prove that if \( f \) is continuous on \([a, b]\), then \(|f|\) is continuous on \([a, b]\). Show by an example that the converse is not true.