Calculus Before Newton and Leibniz: Part II

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Archimedes and Sums of Squares

No description of calculus before Newton and Leibniz could be complete without an account of the contributions of Archimedes, the Greek Sicilian who was born around 287 BC and who died during the Roman siege of Syracuse in 212 BC. We are going to look at one of the many calculus problems he studied.

It is common to credit Archimedes with the earliest stirrings of integral calculus. Among the problems he tackled and solved are finding areas under parabolas and inside spirals and finding the volume of the sphere, spherical segments, and the parabaloid (the solid of revolution obtained by revolving a parabola around its axis). He also showed how to compute the slope of a line tangent to a spiral, the first glimmer of differential calculus.

The beauty of calculus as we now know it comes from its simplicity. The Fundamental Theorem of Calculus enables us to solve very difficult problems by applying simple calculational procedures that are justified by the Fundamental Theorem. Archimedes did not have these, so he had to rely on basic principles and, with a great deal of ingenuity, come up with clever solutions. And he was clever.

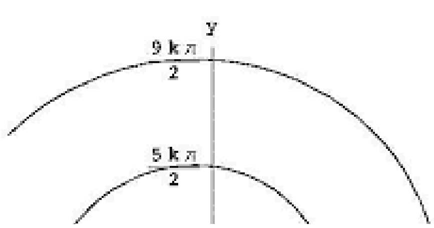
He found volumes of the parabaloid and other solids by using a balancing argument in which he compared the moments of different solids. He calculated the area under a parabola not by the usual method of approximating it by a sum of squares, but by using a geometric observation that enabled him to reduce the problem to one of finding the sum of a geometric series. I'll return to this particular solution in a later piece because it is one of a number of important uses of geometric series that contributed to the development of calculus.

The problem that I want to focus on is that of finding the area inside a spiral. In polar coordinates, the Archimedean spiral is (see Figure 1)

$$r = k\theta$$

for some constant k. In rectangular coordinates, it is the curve with parametric equations

$$x(t) = k \cdot t \cdot \cos t, \; y(t) = k \cdot t \cdot \sin t, \; t >= 0.$$



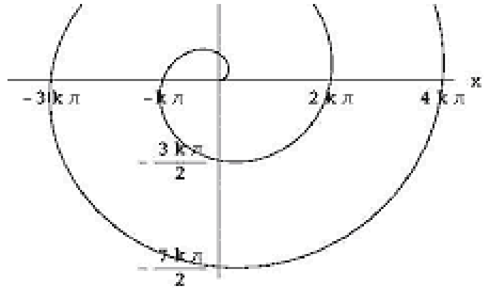


Figure 1: The Archimedean spiral

Archimedes employed what has come to be known as the "method of exhaustion." The idea is to approximate the area using ever smaller pieces whose area can be found exactly, and then to prove that a particular value is the answer by showing that anything smaller is too small and anything larger is too large.

Archimedes attributes this method to Eudoxus of Cnidus (408-355 BC) who proved that the volume of any pyramid or cone is one-third the area of the base multiplied by the height. While Eudoxus found the first proof, this result is even older. Archimedes tells us that it was found by Democritus (*ca.* 465-375 BC). The formula for the volume of a pyramid was also discovered in ancient India, and we have record of it in the Chinese book *Chiu Chang Suan Ching (Nine Chapters on the Mathematical Art)* that may have been written as early as 150 BC.

Even before anyone could prove this formula, finding it required thinking of the pyramid as made up of thin slices, three sets of which could be reconfigured to make a rectangular block. Later in this article, we'll see how this mental three-dimensional geometry led Archimedes to the formula he needed to find the area of the spiral.

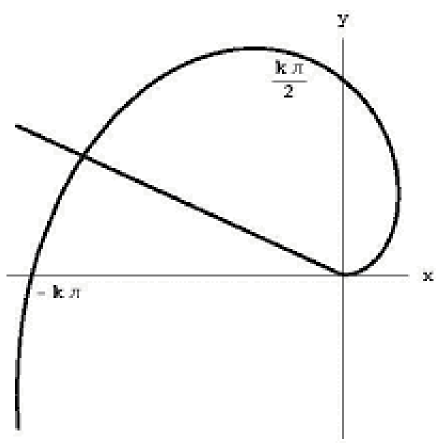


Figure 2: The area bounded by the Archimedean spiral and the ray $\theta = \theta_1$.

Approximating the Area of the Spiral

To find the area bounded by the spiral and the ray $\theta = \theta_1$, as shown in Figure 2, we divide the angle between the rays into n small angles of size θ_1 / n. If we look at the ith piece of angle, the spiral's distance from the origin increases from $k(i - 1)\,\theta_1 / n$ to $ki\,\theta_1 / n$. This means that our area lies inside the sector of radius $ki\,\theta_1 / n$. Our area entirely includes the sector of the circle of radius $k(i - 1)\,\theta_1 / n$.

The angle θ_1 / n represents $\theta_1 / 2\pi n$ of a full circle, so the area of this sector is somewhere between

$$\pi\left(\frac{k(i-1)\,\theta_1}{n}\right)^2 \cdot \frac{\theta_1}{2\pi n} \quad \text{and} \quad \pi\left(\frac{k\,i\,\theta_1}{n}\right)^2 \cdot \frac{\theta_1}{2\pi n}$$

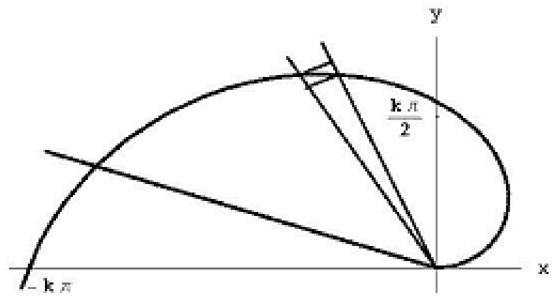


Figure 3: The sector between $\theta = k(i - 1)\theta_1/n$ and $\theta = ki\,\theta_1/n$.

When we simplify these bounds, we see that the area of the ith segment lies between $k^2\theta_1^3 (i - 1)^2 /2n^3$ and $k^2\theta_1^3 i^2 /2n^3$. The total area lies between

$$\frac{k^2\theta_1^3}{2n^3}\left(0^2+1^2+2^2+\cdots+(n-1)^2\right) \quad \text{and} \quad \frac{k^2\theta_1^3}{2n^3}\left(1^2+2^2+3^2+\cdots+n^2\right)$$

At this point, Archimedes derived a succinct formula for the sum of the first $n - 1$ squares:

$$1+4+9+\cdots+(n-1)^2 = \frac{n^3}{3}-\frac{n^2}{2}+\frac{n}{6} \qquad (1)$$

The area under the spiral lies somewhere between

$$\frac{k^2\theta_1^3}{2n^3}\left(\frac{n^3}{3}-\frac{n^2}{2}+\frac{n}{6}\right) = \frac{k^2\theta_1^3}{6}\left(1-\frac{3}{2n}+\frac{1}{n^2}\right)$$

and

$$\frac{k^2\theta_1^3}{2n^3}\left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right) = \frac{k^2\theta_1^3}{6}\left(1+\frac{3}{2n}+\frac{1}{n^2}\right)$$

As Archimedes now argued, the only number that lies between these bounds for all values of n is $k^2\theta_1^3/6$.

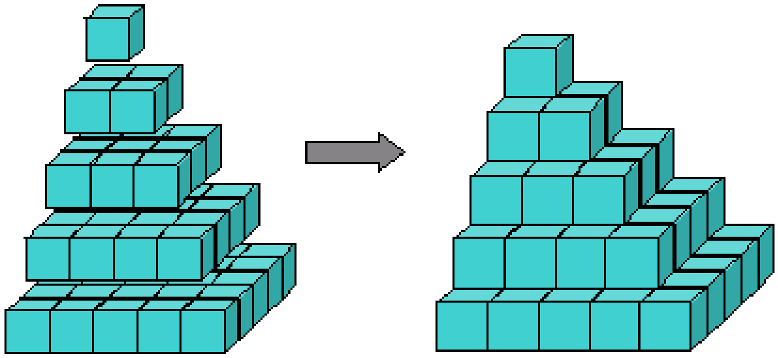


Figure 4: The pyramid of squares

The Formula for the Sum of Squares

The formula for the sum of squares may not have been new to Archimedes, and there is evidence that it might have been discovered about the same time in India. We do know that it was rediscovered many times. The earliest proofs, including Archimedes' proof, are all geometric.

We can visualize the sum of squares as a pyramid built from cubes as in Figure 4. Archimedes showed how to take three of these pyramids together with a triangular layer of blocks representing 1+ 2···+(*n* - 1) and fit them together to get a block of cubes *n x n x (n* - 1). In other words, he showed that

$$3\left(1^2+2^2+\cdots+(n-1)^2\right)+\left(1+2+\cdots(n-1)\right)=n^2(n-1) \qquad (2)$$

.

The formula 1+2+···+(*n* - 1)=*n* (n - 1)/2 is often attributed to Gauss. The story is that he discovered it when his teacher ordered him to add the integers from 1 through 100. In fact, it is an ancient formula. It can be found, for example, in India in a Jain manuscript from 300 BC. It would have been part of Archimedes' own mathematical instruction. It has a very simple geometric proof as shown in Figure 5.

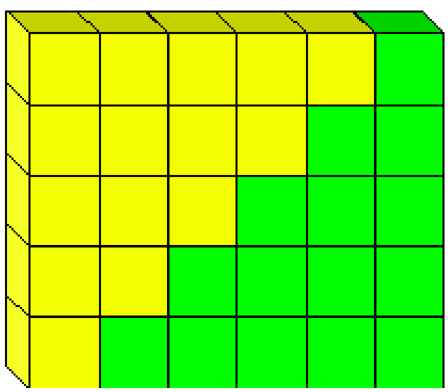


Figure 5: The proof that $2\left[1+2+\cdots+(n-1)\right]=n(n-1)$

We still have to prove equation (2), but once we know it is true, we can combine it with the formula for the sum of the first n - 1 integers to get

$$3\left[1^2+2^2+\cdots+(n-1)^2\right]=n^2(n-1)-(n-1)\frac{n}{2}=n^3-\frac{3}{2}n^2+\frac{n}{2}$$

Divide both sides by 3, and we get equation (1).

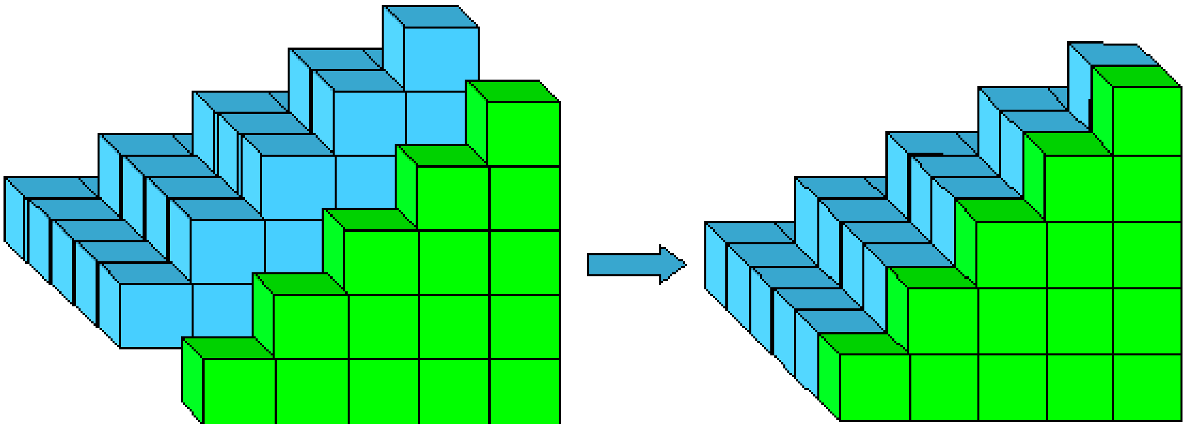


Figure 6: $1^2+2^2+\cdots+(n-1)^2+1+2+\cdots+(n-1)$.

The tricky part of proving equation (2) is to see what you get when you combine a pyramid, $1^2 + 2^2 + \cdots + (n-1)^2$, and the triangular array $1+2+ \cdots + (n-1)$. This is shown in Figure 6, which can be thought of as a pyramid of rectangles:$1\cdot2+2\cdot3+ \cdots + (n-1)^2$. We split this rectangular pyramid into two pieces (Figure 7, pink and blue).

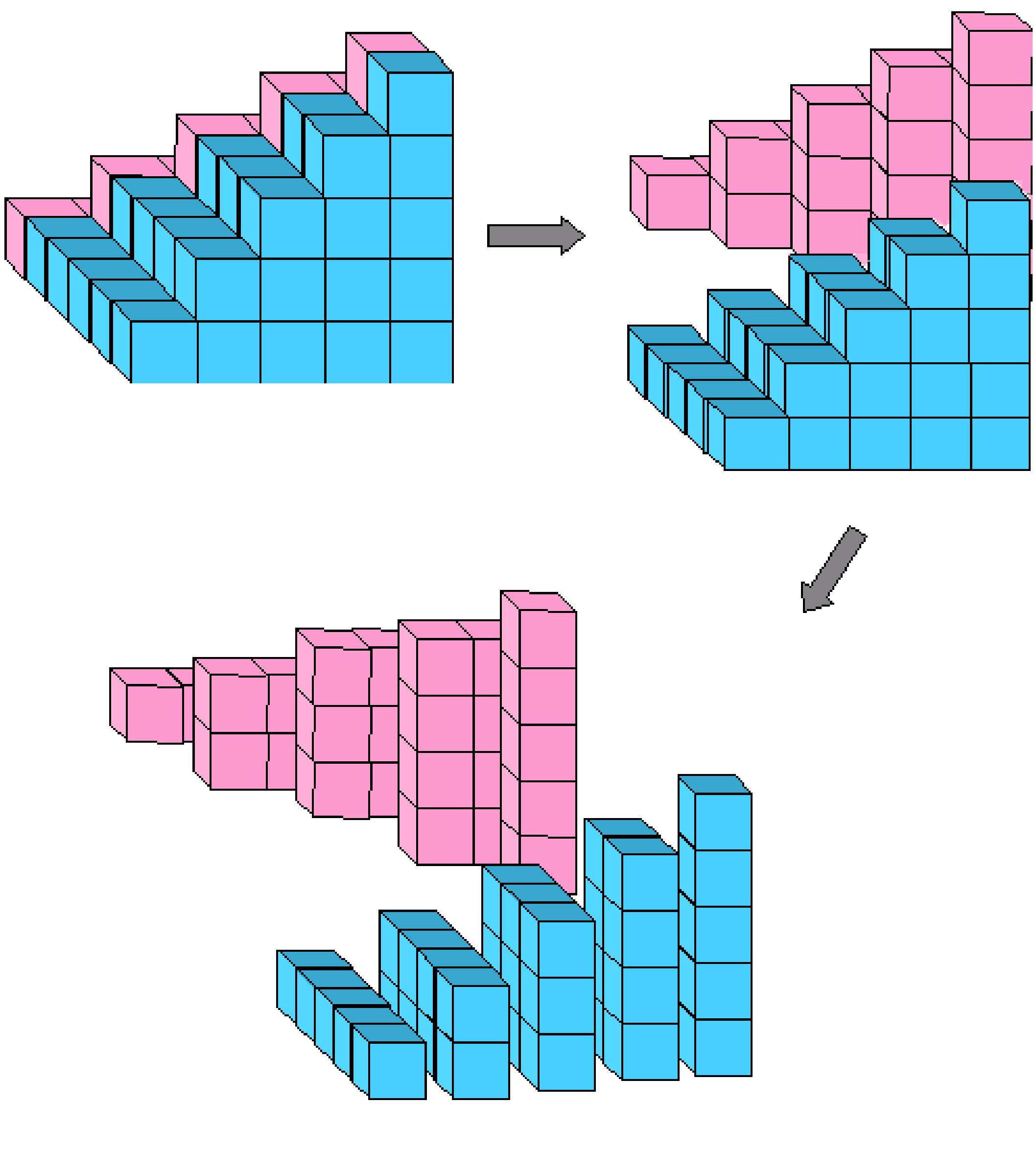


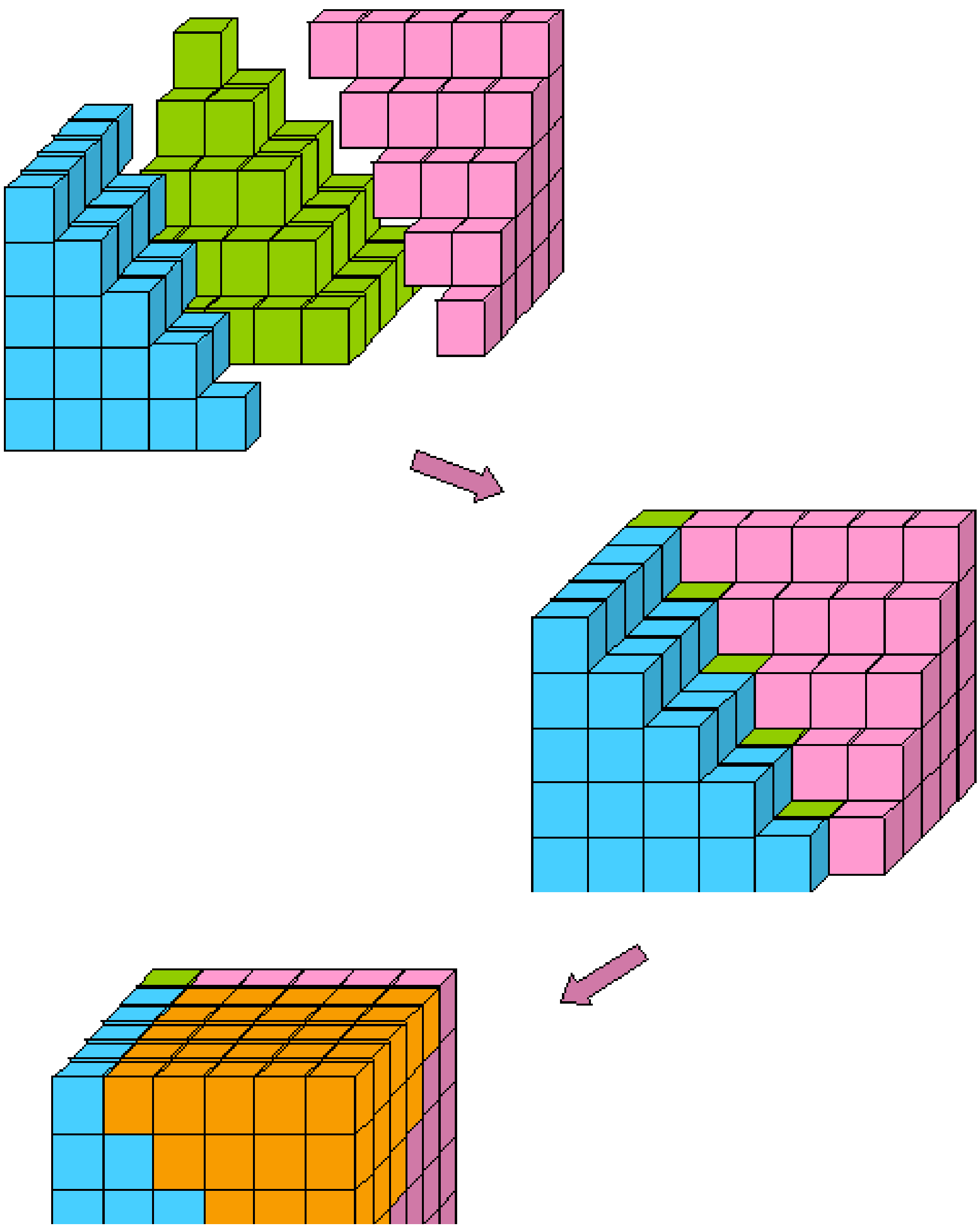
Figure 7: $1\cdot 2+2\cdot 3+3\cdot 4+\cdots+(n-1)n=2\left[1(n-1)+2(n-2)+3(n-3)+\cdots+(n-1)1\right]$

The algebraic equation that corresponds to this picture is

$$\begin{aligned}1\cdot 2+2\cdot 3+3\cdot 4+\cdots+(n-1)n &= 2+(2+4)+(2+4+6)+\cdots+(2+4+\cdots+2(n-1))\\ &= 2\cdot(n-1)+4\cdot(n-2)+6\cdot(n-3)+\cdots+2(n-1)\cdot 1\\ &= 2\left[1(n-1)+2(n-2)+3(n-3)+\cdots+(n-1)1\right]\end{aligned}$$

These pieces fit around the green pyramid in Figure 4 so that one more inverted pyramid completes the *n x n x (n* - 1) block (Figure 8):

$$\sum_{i=1}^{n-1}\left(i^2+2i(n-i)+(n-i)^2\right)=\sum_{i=1}^{n-1}n^2=n^2(n-1)$$



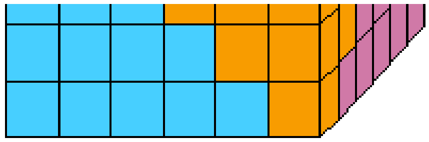


Figure 8: The *n* x *n* x (n - 1) block assembled from three sums of squares and one sum of integers.

Sums of Cubes

You might think that having seen how useful the sum of squares formula is, Archimedes might have then found the formula for the sum of cubes. He didn't. It took over a thousand years before anyone did. The problem is that sums of squares are easy to see geometrically. Sums of cubes can be visualized, but the object you want to put them together to form is four-dimensional. Mathematicians weren't quite ready to tackle the fourth dimension. But as we'll see next time, by the time Europe was in its Middle Ages, mathematicians in the Middle East, India, and China had all done so.

Resources

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