Calculus Before Newton and Leibniz: Part III

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Sums of Powers

In the first two articles in this series, we saw how Archimedes and ibn al-Haytham used formulas for sums of powers to evaluate areas and volumes. Starting in the eleventh century, Arab, Chinese, and Indian mathematicians began to discover techniques that would enable them to find the area under any polynomial. But before anyone could discover such formulas, they had to invent polynomials. Quadratic and cubic polynomials had existed for well over a thousand years, but expressed as areas and volumes. It was not even clear what a fourth power would mean. Higher degree polynomials emerged almost simultaneously around the year 1000 in the Middle East, in India, and in China.

Two eleventh-century contemporaries, Abu Bakr al-Karaji in Baghdad and Jia Xian, a Chinese court eunuch, studied polynomials of high degree, found methods for extracting roots, and discovered what we today call Pascal's triangle. Al-Karaji gave the first known proof of the formula for the sum of cubes. It is also one of the earliest known examples of a complete proof by induction.

Sums of Cubes

When you sum cubes of successive integers, you quickly find a pattern:

$1^3 = 1$, $1^3 + 2^3 = 9$, $1^3 + 2^3 + 3^3 = 36$, $1^3 + 2^3 + 3^3 + 4^3 = 100$, $1^3 + \cdots + 5^3 = 225$.

These are all perfect squares, and not just any perfect squares, but

1^2, $3^2 = (1 + 2)^2$, $6^2 = (1 + 2 + 3)^2$, $10^2 = (1 + 2 + 3 + 4)^2$, $15^2 = (1 + \cdots + 5)^2$.

The formula for the sum of cubes is easy to guess:

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2 = \left(\frac{n(n+1)}{2}\right)^2$$

Aryabhata of Patna in what is now India discovered this formula around 500 CE. It may have been known even earlier. It would be rediscovered by Abu al-Saqr Abd al-Aziz ibn Uthman al-Qabisi in tenth-century Baghdad and again in early fourteenth-century France by Levi ben Gerson, who lived in Provence, near Orange. Nilakantha of Kerala in southwest India gave a visual proof in 1500 that captures the essence of al-Karaji's proof.

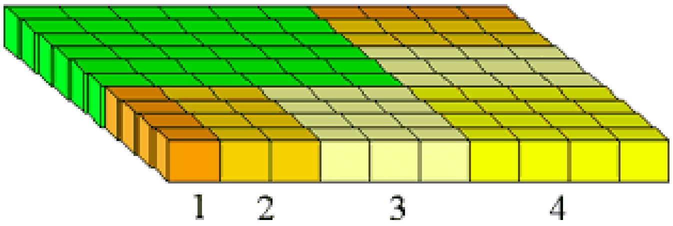

Figure 1: The $(1 + 2 + \cdots + n) X (1 + 2 + \cdots + n)$ square.

Begin with the square $(1 + 2 + \cdots + n)^2$ (Figure 1). We take out the smaller (green) square of size $\big(1 + 2 + \cdots + (n - 1)\big)^2$ and are left with the blocks shown in Figure 2, which can be rearranged into a cube.

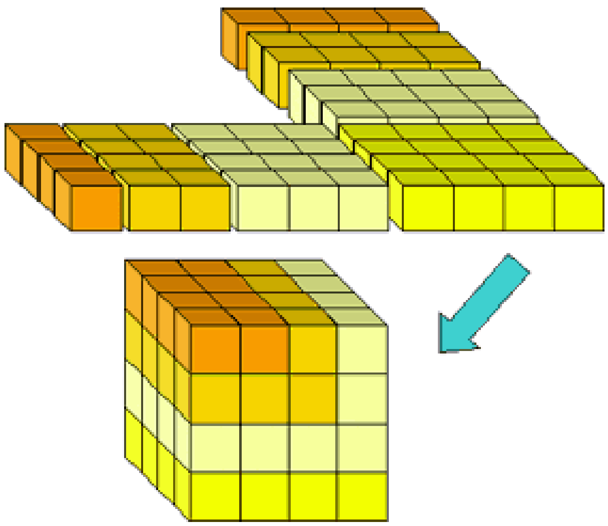

Figure 2: Rearranging the remaining blocks into a cube.

This provides the inductive step for the proof:

$$(1 + 2 + \cdots + n)^2 = n^3 + \big(1 + 2 + \cdots + (n - 1)\big)^2 .$$

Sums of Fourth Powers

In the first article in this series, we saw that ibn al-Haytham needed sums of fourth powers to find the volume of the solid obtained by rotating a parabola around a line perpendicular to the axis of symmetry. Al-Haytham found such a formula. While we have no evidence that he pushed his technique beyond fourth powers, his approach can be used to find formulas for sums of consecutive integers to any power.

Al-Haytham began with the formula for the sum of cubes. He used it to bootstrap up to a formula for sums of fourth powers. This begins with the observation that $k + 1$ can be written as a sum of $k + 1$ 1s or as 2 plus a sum of $k - 1$ 1s or as 3 plus a sum of $k - 2$ 1s, and so on:

$$\big(1^3 + 2^3 + \cdots + (k - 1)^3 + k^3\big)(k + 1)$$

$$= 1^3(1 + 1 + \cdots + 1) + 2^3\,(2 + 1 + \cdots + 1) + \cdots + (k - 1)^3\,\big((k - 1) + 1 + 1\big) + k^3(k + 1)$$

$$= (1^4 + 1^3 + \cdots + 1^3) + (2^4 + 2^3 + \cdots + 2^3) + \cdots + \left((k-1)^4 + (k-1)^3 + (k-1)^3\right) + (k^4 + k^3)$$

We reorder the last sum, putting the fourth powers together and regrouping so that we get sums of consecutive cubes:

$$\left(1^3 + 2^3 + \cdots + (k-1)^3 + k^3\right)(k+1)$$
$$= \left(1^4 + 2^4 + \cdots + (k-1)^4) + k^4\right) + 1^3 + (1^3 + 2^3) + (1^3 + 2^3 + 3^3) + \cdots$$
$$+ \left(1^3 + 2^3 + \cdots + (k-1)^3\right) + (1^3 + 2^3 + \cdots + k^3)$$

We now use the formula for the sum of cubes:

$$\left(1^3 + 2^3 + \cdots + (k-1)^3 + k^3\right)(k+1)$$
$$= \left(1^4 + 2^4 + \cdots + k^4\right) + \sum_{t=1}^{k} \tfrac{1}{4}\left(t^4 + 2t^3 + t^2\right)$$
$$= \left(1^4 + 2^4 + \cdots + k^4\right) + \tfrac{1}{4}\left(1^4 + 2^4 + \cdots + k^4\right) + \tfrac{1}{2}\left(1^3 + 2^3 + \cdots + k^3\right)$$
$$+ \tfrac{1}{4}\left(1^2 + 2^2 + \cdots + k^2\right)$$

We consolidate similar terms and use the formulas for the sums of cubes and squares once more:

$$\tfrac{5}{4}\left(1^4 + 2^4 + \cdots + k^4\right) = \left(\frac{2k+1}{2}\right)\left(1^3 + 2^3 + \cdots + k^3\right) - \tfrac{1}{4}\left(1^2 + 2^2 + \cdots + k^2\right)$$
$$= \left(\frac{2k+1}{2}\right)\left(\frac{k(k+1)}{2}\right)^2 - \frac{1}{4}\left(\frac{k(k+1)(2k+1)}{6}\right),$$
$$1^4 + 2^4 + \cdots + k^4 = \frac{k(k+1)(2k+1)}{10}\left(k(k+1) - \tfrac{1}{3}\right).$$

The same idea will work to find a formula for the sum of fifth powers in terms of the sum of fourth powers, and so on.

Higher powers
In twelfth-century Baghdad and then independently in fourteenth-century India and China, mathematicians discovered and exploited a remarkable property of Pascal's triangle. Starting at the edge, come down any diagonal that heads southwest, adding the entries. Wherever you stop, the sum of these numbers is the next number to the southeast:

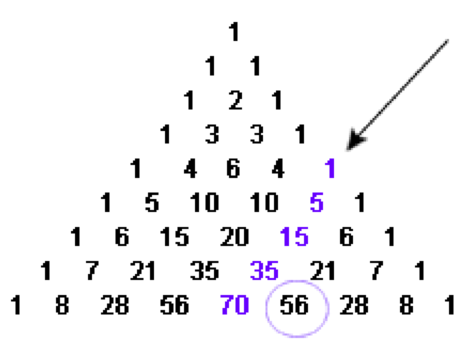

Figure 3: The sum of terms on a southwest diagonal (1+5+15+35) equals

the next term to the southeast (56).

We can express this in terms of polynomials. We define a polynomial of degree *n*:

$$P_n(x) = \binom{x+n-1}{n} = \frac{x(x+1)(x+2)\cdots(x+n-1)}{n!}.$$

This property of diagonal sums implies that for positive integers *n* and *k*, we always have

$$P_n(1) + P_n(2) + \cdots + P_n(k) = P_{n+1}(k).$$

We can use this to find sums of arbitrary powers because any polynomial of degree *n*, including x^n , can be expressed in terms of $P_1(x), P_2(x, \ldots , P_n(x))$. For example,

$$x^4 = 24P_4(x) - 36P_3(x) + 14P_2(x) - P_1(x).$$

It follows that

$$1^4 + 2^4 + \cdots + k^4 = 24P_5(k) - 36P_4(k) + 14P_3(k) - P_2(k).$$

This fundamental relationship for binomial coefficients first appeared in *Al-Bahir fi'l Hisab (Shining Treatise on Calculation)* written by al-Samaw'al in 1144 in what is now Iraq. It also can be found in *Siyuan Yujian (Jade Mirror of the Four Unknowns)* written by Zhu Shijie in 1303 in China, and the *Ganita Kaumudi* written by Narayana Pandita in 1356 in India.

Remarkable as this method is, it is not the best way of finding formulas for the sums of powers. That was discovered by the Swiss mathematician Jacob Bernoulli at the beginning of the eighteenth century. For two thousand years, mathematicians had been using sums of powers formulas in order to calculate areas. Calculus showed us how to do these without relying on sums of powers. Bernoulli saw how to turn the tables. He used integral calculus to find simple derivations of the sum of powers formulas. But that is another story.

Resources

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