Adventures in the Amusement Park
AB2/BC2 from the 2002 AP Calculus Exam

David Bressoud, Macalester College, St. Paul, MN

1. Introduction

2002 AB2/BC2

The rate at which people enter an amusement park on a given day is modeled by the function *E* defined by

$$E(t) = \frac{15600}{\left(t^2 - 24t + 160\right)}.$$

The rate at which people leave the same amusement park is modeled by the function *L* defined by

$$L(t) = \frac{9890}{\left(t^2 - 38t + 370\right)}.$$

Both $E(t)$ and $L(t)$ are measured in people per hour and time t is measured in hours after midnight. These functions are valid for $9 \le t \le 23$, the hours during which the park is open. At time t=9, there are no people in the park.

(a) How many people have entered the park by 5 PM (t=17)? Round answer to the nearest whole number.
(b) The price of admission to the park is $15 until 5:00 PM ($t$=17). After 5:00 PM, the price of admission to the park is $11. How many dollars are collected from admissions to the park on the given day? Round your answer to the nearest whole number.
(c) Let $H(t) = \int_9^t \left(E(x) - L(x)\right)dx$ for $9 \le t \le 23$. The value of $H(17)$ to the nearest whole number is 3725. Find the value of $H'(17)$ and explain the meaning of $H(17)$ and $H'(17)$ in the context of the park.
(d) At what time t, for $9 \le t \le 23$, does the model predict that the number of people in the park is a maximum?

I have followed this problem since its birth. I was a member of the College Board's AP Calculus Development Committee that spent over a year developing and refining it. I served as the Question Leader for it at the 2002 Reading at Colorado State University in Fort Collins, working with others to finalize the standards for scoring student work and explaining those scoring standards to the Readers. And then I took over as chair of the

Calculus Development Committee, thus needing to internalize the lessons learned from this problem that can help guide the development of future exams.

I believe that others, especially AP high school teachers, may be interested in the life of this problem. I will tell the story of 2002: AB2/BC2 from its beginnings through its final version, describe how the scoring standards were set, talk about where students had the most difficulty, and detail how they performed. I will discuss the lessons I have learned and will end with ideas for using this problem as a jumping off point for classroom projects.

If you pay careful attention to AP Calculus exams, you may notice a striking similarity between this question and AB4 from the 2000 exam (shown below). In fact both questions started from the same source. The first time the committee members chose that source, they ran into the difficulties of using continuous functions to model discrete phenomena: people entering and leaving a park. They changed the problem to a continuous flow of water. Since this problem was put on the non-calculator portion of the exam, they simplified the functions so that they could be handled in the fifteen minutes that students would have for it. That common source has since been retired, but both problems worked so nicely that I cannot believe that we have seen the last of free response questions in which students are asked to wrestle with off-setting rates of accumulation.

2000: AB4

Water is pumped into an underground tank at a constant rate of 8 gallons per minute. Water leaks out of the tank at the rate of $\sqrt{t+1}$ gallons per minute, for $0 \le t \le 120$ minutes. At time $t = 0$, the tank contains 30 gallons of water.

(a) How many gallons of water leak out of the tank from time $t = 0$ to $t = 3$ minutes?
(b) How many gallons of water are in the tank at time $t = 3$ minutes?
(c) Write an expression for $A(t)$, the total number of gallons of water in the tank at time t.
(d) At what time t, for $0 \le t \le 120$, is the amount of water in the tank a maximum? Justify your answer.

2. Genesis of the question

The committee met in October of 2000 to begin working on the 2002 exam. We create nine questions for the operational exam (plus another nine for the alternate exam and now a third nine for the Form B exam given overseas). Three of the nine questions cover AB Calculus topics and will appear on both the AB and BC exams, what are called the Common Questions. Three additional questions appear only on the AB exam; three more questions are only for BC.

To help us get started, we have a sheaf of approximately 200 questions that is periodically replenished by asking committee members and former committee members to write problems that have the potential to be worked into exam questions. The committee usually chooses the common questions first, picking interesting questions, and then determines parameters for the remaining questions. We look at what topics and what types of functions are missing. The free response questions do not try to cover the entire syllabus. The multiple-choice portion of the exam is constructed later once it is clear what topics still need to be addressed. But we do try to include a broad representation of topics from the syllabus in the free response questions.

The source of the amusement park problem, taken literally from the original sheaf, is reproduced below. This was our starting point.

The Source

On the average day during the summer months the rate that people enter a particular amusement park at any time t (in hours) is estimated by the function E(t). The rate that they leave is estimated by the function L(t). Both E(t) and L(t) are given in hundreds of people per hour. The park opens at 9:00 a.m. and stays open until 11:00 p.m.

$$E(t) = \frac{9.17}{((x-1)/4)^2 + 1} \qquad L(t) = \frac{8}{((x-8)/3)^2 + 1}$$

(a) Graph y = E(t) and y = L(t) on the axes provided.
(b) To the nearest hundred how many people have entered the park by noon?
(c) Let N(t) represent the number of hundreds of people in the park at any time t. Write an expression for N(t).
(d) At what time (to the nearest minute) is the largest number of people in the park?
(e) If each ticket costs $20, how much revenue is collected on an average day? Express your answer to the nearest dollar.

One of our first decisions was to stay true to the spirit of the question, keeping the amusement park context and functions that would be reasonable for this context. Reciprocals of quadratic polynomials worked nicely for this purpose. That meant that this question would have to be on the calculator portion of the exam. This gave us the opportunity to focus on an exploration of student understanding of basic concepts of calculus. Computation would play a much smaller role than in the 2000 version.

The source problem has some serious flaws that were identified quickly:

- The author had some trouble spelling the variable "t" in the expressions for the functions.
- The units are hundreds of people, and past experience (1996: AB3/BC3 when the units were billions of gallons of cola) showed us that units that are multiples of natural units lead to difficulties.
- The independent variable is measured in hours, but the problem never makes clear how these are to be measured. An analysis of the model suggests that t is measured in hours after 9 AM. But that raised the specter of a host of misunderstandings. It was decided to switch t to hours after midnight so that there would be a closer correspondence between t and the clock time.

- Question (a) is a bad question because it does not involve any calculus. There are times when we will ask a student to use the graphing calculator to produce the graph of a function, but only when it is part of a question that has significant calculus content.
- Questions (b) and (e) are essentially the same question, just slightly different limits and an extra factor of 20 in question (e). We decided to charge different admission prices before and after 5 PM, a decision that looked good at the time, but, as we would come to recognize later in the year, did not address the fundamental problem.

By the end of the October meeting, we had a pretty good idea of the nature of the 2002 free response questions, but there was a lot of work left to do. Every member left with an assignment to either write a question that had been narrowly prescribed, or to revise a question that had been heavily critiqued. AB2/BC2 was in the latter category.

A few weeks before the January meeting we received copies of eighteen complete questions (for the operational and alternate exams). AB2/BC2 appeared in the form shown below.

January 2001

The rate at which people enter an amusement park on a given day is approximated by the function E defined by $E(t) = \dfrac{15000}{t^2 - 24t + 160}$. The rate at which people leave the same amusement park is approximated by the function L defined by $L(t) = \dfrac{9200}{t^2 - 38t + 370}$. Both $E(t)$ and $L(t)$ are measured in people per hour and time t is measured in hours after midnight. These functions are valid from 9:00 AM (t=9), when the park opens, until 11:00 PM (t=23), when the park closes.

(a) How many people have entered the park by noon (t=12)? Round your answer to the nearest whole person.
(b) Let $N(t)$ represent the number of people in the park at time t. Write an expression for $N(t)$.
(c) At what time t is the greatest number of people in the park?
(d) The price of admission to the park is \$20 until 5:00 PM (t=17). After 5:00 PM, the price of admission to the park is \$15. Approximately how many dollars are collected from admission to the park on a given day? Give your answer to the nearest dollar.

We looked carefully at all 18 questions, but AB2/BC2 escaped fairly lightly this time. Many of the questions needed far more work. Some had to be thrown out completely and replacements constructed. AB2/BC2 was looking good by comparison.

A few of the most obvious flaws were identified:

- The numerator 15000 is a problem waiting to happen. Many students would miscount the number of 0's.
- It would be safer to specify the interval over which this model is assumed to be valid, rather than to give just endpoints.
- "Nearest whole person" should be "nearest whole number."

- We cannot ask for an approximate answer without opening ourselves to a range of interpretations of what this means.

In advance of the March meeting, we were sent the version shown below.

March 2001

The rate at which people enter an amusement park on a given day is approximated by the function E defined by $E(t) = \dfrac{15300}{t^2 - 24t + 160}$. The rate at which people leave the same amusement park is approximated by the function L defined by $L(t) = \dfrac{9200}{t^2 - 38t + 370}$. Both *E(t)* and *L(t)* are measured in people per hour and time t is measured in hours after midnight. These functions are valid for $9 \le t \le 23$, the hours during which the park is open.

(a) How many people have entered the park by noon (t=12)? Round your answer to the nearest whole number.
(b) Let *N(t)* represent the number of people in the park at time *t,* for $9 \le t \le 23$. Write an expression for *N(t)*.
(c) At what time *t,* for $9 \le t \le 23$, is the number of people in the park greatest?
(d) The price of admission to the park is \$20 until 5:00 PM (t=17). After 5:00 PM, the price of admission to the park is \$15. According to this model, how many dollars are collected from admission to the park on a given day? Give your answer to the nearest dollar.

The final assignment of points is never made until after the exam has been taken, we have seen several hundred student papers, and we have some appreciation for where there will be difficulties in the grading. But as we create the exam, we make tentative assignments of those nine points for each problem. This was when we began to realize that AB2/BC2 still had serious problems.

We expected that parts a) through c) would each be worth 2 points and part d) would be worth 3. The specific breakdown was

a) 1 point for setup, 1 point for answer
b) 1 point for limits, 1 point for integrand
c) 1 point for identifying the equation to be solved, 1 point for solution
d) 1 point for setup of amount collected before 5 PM, 1 point for setup of amount collected after 5 PM, 1 point for the final answer.

The problem was that we were looking at 5 of the 9 points for setting up definite integrals and two points just for knowing how to use the calculator to evaluate a definite integral. We were testing the same skills repeatedly. While the free response section does not have to test every topic on the syllabus, we do try to avoid testing the same idea or skill repeatedly. We try to keep the number of points riding on a particular skill commensurate with the importance of that skill in the context of a student's total knowledge of calculus.

We considered expanding part c) by asking students to find the maximum number of people, a natural question. But that would simply add another definite integral evaluation. We could ask students to justify their answer, but this ability to justify a maximum or minimum was already being tested in AB1/BC1.

At this point, we threw it open. What other interesting questions could be asked in this context? One favorite was to ask students to determine both the time when the rate at which people are entering is maximized and the time at which the number of people in the park is growing the fastest. Doing this would serve to drive home the distinction between these two rates, a distinction that we know students often have trouble making. After some valiant struggles we abandoned this line. We just couldn't get enough points without writing several additional questions. We can expect students to spend fifteen minutes on this problem, and we were conscious of the fact that it already was long and wordy.

Another question that was considered, however briefly, was to ask the average amount of time that each person spent in the park. Even we on the committee acknowledged that we probably couldn't solve that from a cold start in fifteen minutes. But it is a great question, and I'll talk a bit more about it at the end of this essay.

The decision was finally made to turn part b) around. Instead of asking for an expression for $N(t)$, we would give them the integral expression $H(t) = \int_9^t \left(E(x) - L(x)\right)dx$ and ask them what it meant. We changed the name of the function from *N* to *H* to hide the obvious implication that *N* is a number. We asked students to explain the meaning of $N(12)$ and $N'(12)$.

The version distributed shortly before the July meeting is given below.

July 2001

The rate at which people enter an amusement park on a given day is modeled by the function *E* defined by

$$E(t) = \frac{15300}{t^2 - 24t + 160}.$$

The rate at which people leave the same amusement park is modeled by the function *L* defined by

$$L(t) = \frac{9200}{t^2 - 38t + 370}.$$

Both *E(t)* and *L(t)* are measured in people per hour and time *t* is measured in hours after midnight. These functions are valid for $9 \le t \le 23$, the hours during which the park is open.

(a) How many people have entered the park by noon (t=12)? Round your answer to the nearest whole number.

(b) Let $H(t) = \int_9^t \left(E(x) - L(x)\right)dx$ for $9 \le t \le 23$. Explain the meaning of $H(12)$ and $H'(12)$ in the context of the amusement park.

(c) At what time *t,* for $9 \le t \le 23$, is the number of people in the park greatest?

(d) The price of admission to the park is \$20 until 5:00 PM (t=17). After 5:00 PM, the price of admission to the park is \$15. According to this model, how many dollars are collected from admissions to the park on a given day? Give your answer to the nearest dollar.

Between March and July, the 2001 exam was administered and read. A major controversy of that reading had involved 2001: AB2/BC2 which gave a function $P(t)$, the temperature in a pond at time t, and asked the student to find the value of $P'(12)$ and to "explain the meaning of your answer in terms of water temperature." The committee's intention in asking this question was to see whether the students could recognize that the fact that this value was negative implied that the water temperature was decreasing. Many students failed to draw this implication. A large number of them said that the number they had found was the rate of change of the water temperature. Looking back, we could see that the question was ambiguous.

When the Chief Reader, Exam Leaders, Question Leaders, and Table Leaders met in the first week of June 2001 to decide how the 2001 exam would be scored, the acceptability of this common answer was hotly debated. Those arguing in favor of allowing this answer pointed out that it is a correct statement about $P'(12)$. Those arguing against accepting it pointed out that it is a formulaic response that students can use without any understanding of what it means. The question specifically asks for meaning. Eventually, and especially because it accorded with what the committee was trying to test, it was decided that the students who only responded that their answer was the rate of change of temperature at time $t = 12$ would not receive credit.

It is worth noting that a large number of students answering AB2/BC2 in the 2002 exam correctly described $H'(17)$ as the rate of change of the number of people in the park at 5 PM. But they lost that point when they added that it gives the rate at which people are leaving the park. It is clear that many students "know" that the derivative is the rate of change without understanding what that means.

When the committee arrived for the July meeting, we could see that we would have to revisit what was then part b). Were we again looking for the student to recognize that the positive derivative meant that the number of people in the park was increasing? If that was our intent, could we find a clearer way of expressing this question?

We spent some time on the wording of the question, discovering how difficult it is to make it more precise without giving away much of the information we wanted the student to produce. We finally realized that there was a fundamental difference between asking for the meaning of $H'(12)$ in the 2002 exam and the meaning of $P'(12)$ in the 2001 exam. In the 2001 exam, we told them that $P(t)$ was the temperature at time t. In the 2002 exam, the students had to discover for themselves the meaning of $H(t)$. We decided that for this question, stating that $H'(12)$ was the rate at which the number of people in the park was changing at noon ($t = 12$) required some insight and would get credit. In line with our original intention, we would also accept the interpretation of the fact that this number is positive: the number of people in the park is increasing.

But our problems were not over. We were still looking at 2 points for parts a), b), and c), and 3 points for part d). We still had two points for using the graphing calculator to evaluate a definite integral that had already been set up. This was too much. We eventually realized that part a) was easily worth 3 points: We were not giving students the limits of that integral; students had to dig them out of the problem context. Just finding those limits was worth 1 point. Correctly identifying the function to be integrated was worth a second point. The third point would be for finding the value of that integral. We could now tie the old part d) to part a) by changing the time in part a) to 5 PM. We assigned only one point to the new part b) because it was testing only one concept: does the student know how to use definite integrals of the rate at which people are entering the park to calculate the amount of money collected in one day? The lost point was made up by also asking for the value of $H'(17)$. To avoid students unnecessarily calculating $H(17)$, we gave them that value.

Tying together the new parts a) and b) had the beneficial effect of enabling students to use the answer from part a) in part b), but it also had an unanticipated consequence. Until now, we had expected that the student who calculated the amount of money collected in one day would do all of the evaluations and calculations *before* rounding to the nearest dollar. This is a model, not an exact description of what happened on that day. One needs to work through the predictions implied by the model, keeping as much accuracy as possible. At the end of the calculations is when to ask how many of the digits are relevant.

But in part a) we had asked students to evaluate $\int_9^{17} E(t)\,dt$ rounded to the nearest integer, thus inviting them to multiply their answer by \$20 to find the total amount collected before 5 PM. A student might well assume that $\int_{17}^{23} E(t)\,dt$ should also be rounded to the nearest integer before multiplying by \$15. The committee realized that this set up two equally valid ways of answering the new question b).

In fact, we had opened ourselves to a wide assortment of rounding strategies that, it could be argued, were all equally valid. This leads me to reflect on the issue of how we handle rounding in the AP Calculus exams. No one particularly likes the arbitrary rule that all decimal answers are to be reported accurate to three places to the right of the decimal. But it is easy to explain what that means, and we can penalize students who fail to maintain sufficient accuracy in their calculations. Most of the time, it is not difficult to devise questions for which it is reasonable to request three digits to the right of the decimal. When modeling a problem that involves tens of thousands of people, it would be ridiculous to request such accuracy. We asked for answers to the nearest integer. Because our expectations were less clearly established, we were forced to greater flexibility in what we would accept. In particular, there was no clear understanding even among the teachers whether the number of people should be rounded to the nearest integer in intermediate calculations.

We changed the admission prices. For the students who used the exact values of the definite integrals, multiplied by the price of admission, they might round their answers before adding the values or round after adding. The new prices were constructed so that the answer would be the same in either case. The functions were also modified for the last time so that at 11 PM there was exactly 1 person left in the park (to turn out the lights and lock up?).

We also realized that we had never specified the initial value. How many people were already in the park when it opened at 9 AM? We corrected that omission.

The last inspired change was to put parentheses around the expression in the denominator. This did not completely eliminate misentering the integral as $\int_9^{17}\left(15600/t^2-24t+160\right)dt$, but it helped to make this an uncommon error.

October 2001

The rate at which people enter an amusement park on a given day is modeled by the function *E* defined by

$$E(t)=\frac{15600}{\left(t^2-24t+160\right)}.$$

The rate at which people leave the same amusement park is modeled by the function *L* defined by

$$L(t)=\frac{9890}{\left(t^2-38t+370\right)}.$$

Both *E(t)* and *L(t)* are measured in people per hour and time *t* is measured in hours after midnight. These functions are valid for $9\le t\le 23$, the hours during which the park is open. At time *t*=9, there are no people in the park.

(a) How many people have entered the park by 5 PM (*t*=17)? Round your answer to the nearest whole number.
(b) The price of admission to the park is \$15 until 5:00 PM (*t*=17). After 5:00 PM, the price of admission to the park is \$11. How many dollars are collected from admissions to the park on the given day? Round your answer to the nearest whole number.
(c) Let $H(t)=\int_9^t\left(E(x)-L(x)\right)dx$ for $9\le t\le 23$. The value of $H(17)$ to the nearest whole number is 3725. Find the value of $H'(17)$ and explain the meaning of $H(17)$ and $H'(17)$ in the context of the amusement park.
(d) At what time *t,* for $9\le t\le 23$, does the model predict that the number of people in the park is a maximum?

In October 2001 the committee saw this problem for the last time before the exam was administered. AB2/BC2 was now in page proof form. Two words were deleted; each of which shortened the text by one line.

3. Scoring the question

The 2002 exam was given in May, and the committee eagerly followed the AP Calculus listserve to gauge teacher response to the exam. It was favorable. People thought it was an interesting exam that was asking the right kinds of questions, but there were complaints that it was long and wordy and required a lot of calculator activity in the first three questions of the free response section. A lively discussion ensued over whether \$104,041 or \$104,048 was the correct response to part b). The former answer is what you get if you round to the nearest integer the number of people who enter before 5 PM and the number who enter after 5 PM, and then calculate the money collected. The latter answer results from doing all calculations with full accuracy and rounding at the end.

A week before the Reading began, the Chief Reader, Exam Leaders, and Question Leaders gathered in Fort Collins to determine point distributions and to identify the big questions that would need to be decided. I was present as Question Leader for AB2/BC2. Craig Wright and Fred Kleumpen of ETS had gone though a few hundred answer books and pulled about fifty "interesting" responses for each question. The answers for AB2/BC2 revealed no significant surprises. We now formally adopted the point distribution that the Chief Reader had proposed.

AB2/BC2 Solutions

a) 3 points: 2 points for setting up the integral (1 for the correct limits and 1 for the correct integrand) and 1 point for the answer.

$$\int_9^{17} E(t)\,dt = 6004 \text{ people.}$$

b) 1 point:

$$\$15 \times \int_9^{17} E(t)\,dt + \$11 \times \int_{17}^{23} E(t)\,dt = \$104048$$

or

$$\$15 \times 6004 + \$11 \times 1271 = \$104041.$$

c) 3 points: 1 point for the value of $H'(17)$ and 1 point for each of the explanations.

$$H'(17) = E(17) - L(17) = -380.281 \text{ people/hour}$$

$H(17)$ is the number of people in the park at time $t = 17$. $H'(17)$ is the rate at which the number of people in the park is changing. The number of people in the park is decreasing at a rate of approximately 380 people per hour.

d) 2 points: 1 point for recognizing that this occurs when $E(t) - L(t) = 0$ and 1 point for finding the answer, $t = 15.794$ or $t = 15.795$ hours after midnight.

Two days later, the Table Leaders arrived. I began working with Steve Greenfield from Rutgers University and Al Eichelberger from University High School in Tucson, Arizona to decide what constituted an acceptable answer for each of the nine points. We prepared to brief the Readers, those grading this problem, on how to assign these points. The three of us read many hundreds of student solutions and debated what we would accept. We also got feedback from the Table Leaders, 89 experienced readers who help to refine the scoring standards and assist the newer readers. The final scoring standard was discussed with and approved by Larry Riddle of Agnes Scott College, the Chief Reader, Caren Deifenderfer of Hollins University, the AB Exam Leader, and Mike Boardman of Pacific University, the BC Exam Leader.

The main issues involved parts b) and c). Much computation goes into part b), but it only counts for 1 point. Since part a) was already testing the ability to evaluate a definite integral on the graphing calculator, we decided that the student who got as far as

$$\$15 \cdot \int_9^{17} E(t)\,dt + \$11 \cdot \int_{17}^{23} E(t)\,dt$$

had shown enough to earn this point. We would ignore anything that came after this. This had the added benefit that it neatly sidestepped the problem of how to handle the many rounding strategies that we were seeing.

Many teachers at the Reading complained that if the setup was all the students needed to write to have the answer counted as correct, then we should not have asked for how much money actually was collected. When the question was first written, the committee had expected that evaluation of the integrals would be worth a point. In retrospect, once we made the tentative assignment of only one point to this part of the problem, we should have rewritten it to only ask for the setup.

As we read student work, we saw that there was a lot of confusion in part c) about rounding. We had asked students to round their answers to the nearest integer in parts a) and b), and the first value in part c) was rounded to the nearest integer. Even though the value of $H'(17)$ is a rate, and thus it is appropriate to report it as a decimal, many students assumed that we also wanted them to round this answer to the nearest integer. We decided that for this particular answer, we would relax our usual rules about reporting decimal answers.

The other problem in part c) was what to accept as an explanation of "the meaning of $H(17)$ and $H'(17)$ in the context of the park." One piece of that explanation had to be an indication that these numbers were describing something that was happening at $t = 17$ or 5 PM. We saw students who were careful about this, students who ignored the value of t completely, and students who made a single reference to that time. The last of these was often in a manner in which it was difficult to tell whether they intended the time reference to apply to both explanations or only to one of them.

With two points available for these two explanations, we decided that the student who failed to make any reference to $t = 17$ or 5 PM would lose one point, provided at least one point would otherwise be earned. A single reference to the time in either explanation would be sufficient.

The meaning of $H(17)$ is fairly straightforward. It is either the number of people in the park at 5 PM or it is the difference between the number of people who enter before 5 PM and the number who leave before 5 PM. The meaning of $H'(17)$ has more variability.

The committee had used the phrase "meaning of $H'(17)$" intentionally. We did not want to give hints by being more specific. As the discussions in July had made clear, there were two types of acceptable answers. Now preparing for the Reading, we classified these as *interpretation*: "It is the rate at which the number of people in the park is changing at 5 PM," and *implication*: "Since $H'(17) = -380.281 < 0$, the number of people in the park at 5 PM is decreasing."

The situation was further complicated by the fact that students often gave several different meanings, and a lot of what they wrote was either incomplete or ambiguous. This included statements such as:

- the rate of the number of people at 5 PM.
- the rate at which people enter/leave (enter and leave, enter or leave) at 5 PM.
- the rate at which people stay at 5 PM.
- this compares the rates at which people enter and leave at 5 PM.

It is significant that all these students knew it was a rate. What they had trouble identifying was how to connect "rate" and "number of people" in a meaningful way. We decided to classify these statements as *ambiguous*. *Acceptable* interpretations included

- the rate of change of the number of people at 5 PM.
- the rate at which the number of people is changing at 5 PM.
- the net rate of change of the number of people at 5 PM.
- the difference between the rate at which people are entering and the rate at which they are leaving at 5 PM.

Unacceptable interpretations included

- any reference to an average rate of change.
- any reference to something that occurs over an interval of time.
- the use of 380 as a number of people (rather than as a rate in people/hour): "At 5 PM, 380 people are leaving the park."
- the description of this number as the rate at which people leave the park.

To earn this point, the student would have to give at least one acceptable explanation and no unacceptable explanations. Ambiguous explanations would, in effect, be ignored. It was discouraging to see how many students correctly identified $H'(17)$ as the rate of change of the number of people in the park, and then went on to lose the point by trying to clarify the meaning and saying something like "This means that at 5 PM, people are leaving the park at the rate of 380 people per hour."

Most students did best on parts a) and b) and on the explanation of $H(17)$. The integral as accumulator seems to be a concept that is well understood. Even though students should be familiar with the Fundamental Theorem of Calculus, it entails another level of difficulty to recognize that this means that the rate of change of the accumulator is the original function. The value and meaning of $H'(17)$ were difficult for many students. The difficulty level was still higher in part d). First, students had to realize that to answer this question they had to maximize $H(t)$ which required finding where $H'(t) = 0$. By itself, this earned no points. The next step was to realize that the Fundamental Theorem of Calculus implies that $H'(t) = 0$ when $E(t) - L(t) = 0$. Only now did they earn the first point for part d). Unfortunately, many students only got to this point after they had put away their graphing calculators. While solving this equation can be reduced to finding the roots of a quadratic polynomial, the numbers are sufficiently nasty that very few students successfully found the value of t without a calculator.

Scores

The average score for this problem on the AB exam was 3.13 with a standard deviation of 2.82. The average score on the BC exam was 4.90 with a standard deviation of 2.81. The tables reflect how students at each level of performance did on this question. The first column gives the final grade on the AP exam (3 is qualified, 4 is well qualified, 5 is extremely well qualified). The second column gives the percentage of students who earned that grade. The third column gives the range of scores of students who were in that percentile range on this problem (for example, students in the top 18% on this question in the AB exam scored at least 6 points, students in the next 23% scored 4, 5, or 6 points). The last column is the average score of students in this percentile range for this question.

AB exam

Grade	Percentage	Range of score	Average score
5	18.0%	6-9	7.39
4	23.0%	4-6	5.00
3	26.3%	1-4	2.43

BC exam

Grade	Percentage	Range of score	Average score
5	43.3%	6-9	7.46
4	16.3%	5-6	5.30
3	21.6%	2-5	3.40

4. Looking back/looking ahead

There were a number of things we learned from the reading of this problem that will influence future problems.

- Students need to know how to use their calculators effectively. Most students would have used the calculator four times on this problem: to evaluate the integral in part a), to evaluate the integral or integrals in part b), to evaluate $E(17)-L(17)$ in part c), and to solve the equation $E(t)=L(t)$ in part d). If the student knew to store the given functions into the calculator before even beginning this problem, then those are not unreasonable calculations to request. A student who re-entered those functions each time they were needed lost several of the fifteen minutes available for thinking about the problem. Repeatedly re-entering the functions also introduced additional opportunities for error. The word needs to get out to AP classes that for the free response questions, students should define in their calculators the functions that are given before they begin to answer the questions.
- Changing the requirements for rounding leads to a lot of confusion. If we want anything other than three digits to the right of the decimal, we either need to be very precise in what we want and when we want it, or we need to be flexible in how we are willing to read answers.
- When explaining the meanings of $H(17)$ and $H'(17)$, many students neglected to refer to the time $t=17$ (or to 5 PM). This important part of the meaning was ignored too often.
- The fact that so many students were at a loss in part d), even though most of them did realize that $H(t)$ was the number of people in the park at time *t*, tells us that they do not understand the Fundamental Theorem of Calculus as completely as they should. It is a signal that we need to ask more questions like this.

- Students need to be better prepared for what will be expected on maxima/minima questions. On the one hand, using a maximum or minimum key on a graphing calculator is never sufficient for the exam. We always want to see calculus being used to find where these values occur. On the other hand, justification that a maximum or a minimum actually occurs at the designated point is not necessary unless it is specifically requested. There were students who lost valuable time in part d) in trying to establish that the critical point at $t = 15.795$ actually was a local maximum rather than a local minimum.

The most common complaint about AB2/BC2 was that it was too wordy. It was. It presented students with a lot to read and digest. The nature of what we were attempting forced this on us. When asking unfamiliar questions, we cannot afford ambiguity or openings for misunderstanding. That means adding more words. The calculator is a wonderful tool that enables us to handle functions that make sense for the problem at hand, but they also eliminate many of the points that would be allocated for demonstrating knowledge of techniques of integration or differentiation. Losing these points means that more questions need to be asked.

On the whole, this problem worked well. It's an interesting problem that gets at fundamental ideas of calculus, and it produced the even spread of scores that make it easy to delineate between the five possible total grades. Every part required some interpretation. None of it was purely formulaic. And students, for the most part, did well. A score of five out of nine under the time constraints and pressures of this exam is a respectable showing. 35% of the AB students and 61% of the BC students scored at least five out of nine points on this question.

5. Further explorations

We only asked a few of the many questions that we could have posed. Here are some others. The first two get at the confusion many students had between the rate at which people enter and the rate at which the number of people in the park increases. The next two are obvious questions that we wish we could have asked, but they didn't test anything that wasn't already being tested. Question (e) is a problem that shows how moments enter naturally into this context. Question (f) provides a challenging set of maximization problems and opens the door to exploration of functions of several variables.

(a) When is the rate at which people are entering the amusement park the greatest?
Ans. 12 noon

(b) At what time is the total number of people in the park increasing most quickly?
Ans. Between 11:41 and 11:42 AM ($t = 11.6918$)

(c) What is the greatest number of number of people in the park at any time?
Ans. 3951

(d) How many people are in the park when it closes at 11 PM?
Ans. 1

(e) What is the average amount of time each person spends in the park?
Ans.:

Start with a simpler question. If people enter the park at the rate given by $E(t)$ but everyone stays until 11 PM ($t = 23$), what is the average amount of time each person spends in the park?

If we pick a time, say t_i, then the number of people who enter the park near this time is given by $E(t_i)\Delta t$. Actually, this is an approximation to the number of people who enter between $t_i - \Delta t/2$ and $t_i + \Delta t/2$. Each of these people stays in the park $23 - t_i$ hours. For our simpler problem, the total number of people hours spent in the park is

$$\lim_{\Delta t \to 0} \sum E(t_i)\Delta t(23 - t_i) = \int_9^{23} E(t)(23 - t)\,dt.$$

We now go back to the original question. If someone leaves at 3om, aht decreases the total person hours spent in the park by 8. The total number of person hours spent in the park is

$$\int_9^{23} E(t)(23 - t)\,dt - \int_9^{23} L(t)(23 - t)\,dt = 30382.933.$$

The total number of people who visit the park is

$$\int_9^{23} E(t)\,dt = 7275.553.$$

The average amount of time spent in the park is

$$\frac{30382.933}{7275.553} = 4.176 \text{ hours } = 4 \text{ hours, } 11 \text{ minutes.}$$

(f) A new model for the rate at which people enter the park at a different part of the summer season takes account of the price p in dollars that is charged for admission. This new model for the rate at which people enter the park is given by

$$F_p(t) = \frac{\left(266000 - 770p - t\,p^3\right)}{\left(10t^2 - 240t + 1600\right)}.$$

I. Find how much money the park would collect from the entrance fees if they charge $p = \$15$ before 5 PM ($t = 17$) and $p = \$11$ after 5 PM.

(Answer: $143039.)

II. If the park decides to charge the same price all day long, what price would maximize the total collected from entrance fees? How much would the park collect in this case?

(Answer: $p = \$16.33$; $147162. The total amount collected at price p is

$$p\int_9^{23} F_p(t)\,dt = p\int_9^{23} \frac{266000}{10t^2 - 240t + 1600}\,dt - p^2\int_9^{23} \frac{770}{10t^2 - 240t + 1600}\,dt$$
$$- p^4\int_9^{23} \frac{t}{10t^2 - 240t + 1600}\,dt$$

Evaluate the three definite integrals and then find the maximum value of the resulting fourth degree polynomial.

III. If the park decides to charge one price before 5 PM and a different price after 5 PM, what prices should they charge in order to maximize the total collected from entrance fees? How much will they collect? How much money does the park lose if it rounds these prices to the nearest half-dollar?

(Answer: $16.81 before 5, $14.64 after 5. $147999. If we round prices to $17 and $14.50, the park collects $147963, so it loses $36.)

IV. There is no reason why prices have to change at time $t = 17$. We can introduce a new variable, *x*, and have the prices change when $t = x$. Using the prices that you found in the last question (rounded to the nearest half-dollar), find the time (to the nearest tenth of an hour) at which to change price that will maximize the total amount collected from entrance fees. How much will be collected?

(Answer: You want to maximize

$$17\int_9^{x} F_{17}(t)\,dt + 14.50\int_x^{23} F_{14.5}(t)\,dt.$$

The derivative of this function (with respect to *x*) is $17\,F_{17}(x) - 14.5\,F_{14.5}(x)$ which is zero when $x = 15.4$ (3:24 PM). $148109 is collected.)

V. Using the time find in the last question, adjust the prices in increments of $0.10 and the time in increments of 0.1 hours until you find values that maximize the total amount of money collected. Then adjust the prices in increments of $0.01 and the time in increments of 0.01 hours. To the nearest dollar, what's the maximum amount that can be collected?

(Answer: $148246. We get this by charging $17.25 before 2 PM and $15.25 after 2 PM.)