The Joys of Mathematics with Doron

David Bressoud
Macalester College
St. Paul, MN

From $A = B$ to $Z = 60$
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PowerPoint available at
www.macalester.edu/~bressoud/talks
Dear Dan,

Thanks a lot for your
regards. I was particularly impressed and
very much enjoyed the easy proof of
the R-R identities. Did you move
to Wisconsin or are you going back to
Penn State? We are a direct co-limit of
the Cauchy 18 (p.1.6) and this book on
partition:

\[ P(\infty) + P(\infty) + P(\infty) + \ldots \]

The \( \ell_2 \) space forms parts of
partition (1,0) when it is a regular partition and \( \ell_1 \)
a certain partition \( \ell_1 \) partition, \( \ell_1 \mid \ell_1 = \infty \) or \( \ell_1 \)

If the \( \ell_1 \) is the part of the \( \ell_1 \) space partition
not \( \ell_1 \) in the block for \( \ell_1 \) of

Else could get a catch block
POLL and KAR
Euler’s Theorem:

\[
\prod_{n=1}^{\infty} (1 - q^n) = \sum_{k=-\infty}^{\infty} (-1)^k q^{3/2 k^2 + \frac{1}{2} k}
\]

Combinatorial interpretation:

\# of partitions of \( n \) into even \# distinct parts minus \# of partitions of \( n \) into odd \# distinct parts = \((-1)^k\), if \( n \) is a pentagonal number \((n = \frac{3}{2} k^2 + \frac{1}{2} k)\)

0, if not

There is a simple, bijective proof of this due to Fabian Franklin.
Equivalent Theorem:

\[ 1 = \frac{1}{\prod_{n=1}^{\infty} (1 - q^n)^k} \sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{3}{2}k^2 + \frac{1}{2}k} \]

Combinatorial interpretation: For \( n \geq 1 \) and \( k \) in \( \mathbb{Z} \),

\[ \sum_{k \text{ even}} p\left(n - \frac{3}{2}k^2 - \frac{1}{2}k\right) = \sum_{k \text{ odd}} p\left(n - \frac{3}{2}k^2 - \frac{1}{2}k\right) \]

Garsia and Milne had used their involution principle to turn Franklin’s proof into a bijection between the two sets represented by the sides of this equality.

Doron’s question: Is there a more natural bijection?
\[ \sum p\left(n - \frac{3}{2}k^2 - \frac{1}{2}k\right) \]

This sum counts pairs \((\pi, k)\), where \(\pi\) is a partition of \(n - \frac{3}{2}k^2 - \frac{1}{2}k\) and \(k\) can be any integer; positive, negative, or zero.


\[ t = \# \text{ of parts in } \pi, \quad l = \text{largest part in } \pi \]

If \(l - t \leq 3k\), subtract 1 from each part, construct new largest part of size \(t + 3k - 1\), \(k\) becomes \(k - 1\).

If \(l - t > 3k\), remove largest part, add 1 to \(l - 3k - 2\) parts, some of which may be 0, \(k\) becomes \(k + 1\).
Doron’s next challenge:

Let’s produce a bijective proof of the Rogers-Ramanujan identities.


B., Easy Proof of the Rogers-Ramanujan Identities,  
\textit{J. Number Theory}, 1983

\[ (-xq; q)_s (-x^{-1}; q)_s = \sum_m x^m q^{(m^2 + m)/2} \begin{bmatrix} 2s \\ s - m \end{bmatrix} \]

\[ \frac{1}{(q; q)_{n+j}} = \sum_{i \geq 0} \frac{q^{i^2 + 2ij}}{(q; q)_{i+2j}} \begin{bmatrix} n - j \\ i \end{bmatrix} \]

\[ \Rightarrow \sum_j \frac{x^j q^{aj^2}}{(q; q)_{n-j} (q; q)_{n+j}} = \sum_{s \geq 0} \frac{q^{s^2}}{(q; q)_{n-s}} \sum_j \frac{x^j q^{(a-1)j^2}}{(q; q)_{s-j} (q; q)_{s+j}}, \]

\[ \frac{1}{(q; q)_\infty} \sum_j x^j q^{aj^2} = \sum_{s \geq 0} q^{s^2} \sum_j \frac{x^j q^{(a-1)j^2}}{(q; q)_{s-j} (q; q)_{s+j}} \]
\[
\frac{(-xq^3; q^5)_\infty}{(q; q)_\infty} \frac{(-x^{-1}q^2; q^5)_\infty}{(q; q)_\infty} (q^5; q^5)_\infty = \frac{1}{(q; q)_\infty} \sum_j x^j q^{(5j^2 + j)/2}
\]
\[
\frac{(-xq^3; q^5)_\infty (-x^{-1}q^2; q^5)_\infty (q^5; q^5)_\infty}{(q; q)_\infty} = \frac{1}{(q; q)_\infty} \sum_j x^j q^{(5j^2 + j)/2} = \sum_{s \geq 0} q^{s^2} \sum_j \frac{x^j q^{(3j^2 + j)/2}}{(q; q)_{s-j} (q; q)_{s+j}}
\]
\[
\frac{(-xq^3; q^5)_\infty (-x^{-1}q^2; q^5)_\infty (q^5; q^5)_\infty}{(q; q)_\infty} = \frac{1}{(q; q)_\infty} \sum_j x^j q^{(5j^2+j)/2}
\]

\[
= \sum_{s \geq 0} q^{s^2} \sum_j x^j q^{(3j^2+j)/2} \frac{1}{(q; q)_{s-j} (q; q)_{s+j}}
\]

\[
= \sum_{s, t \geq 0} \frac{q^{s^2+t^2}}{(q; q)_{s-t}} \sum_j x^j q^{(j^2+j)/2} \frac{1}{(q; q)_{t-j} (q; q)_{t+j}}
\]
Rogers-Ramanujan identities are $x = -1$ and $x = -q$. 

\[
\frac{(-xq^3; q^5)_\infty}{(q; q)_\infty} \frac{(-x^{-1}q^2; q^5)_\infty}{(q; q)_\infty} \frac{(q^5; q^5)_\infty}{(q; q)_\infty} = \frac{1}{(q; q)_\infty} \sum_j x^j q^{(5j^2 + j)/2} \\
= \sum_{s \geq 0} q^{s^2} \sum_j x^j q^{(3j^2 + j)/2} \frac{1}{(q; q)_{s-j} (q; q)_{s+j}} \\
= \sum_{s, t \geq 0} q^{s^2 + t^2} \sum_j x^j q^{(j^2 + j)/2} \frac{1}{(q; q)_{s-t} (q; q)_{t-j} (q; q)_{t+j}} \\
= \sum_{s, t \geq 0} \frac{q^{s^2 + t^2}}{(q; q)_{s-t} (q; q)_{2t}} (-xq; q)_t (-x^{-1}; q)_t
\]

1983: Received preprint of Doron’s proof of Andrews’ conjecture (a.k.a. $q$-Dyson),

$$\prod_{1 \leq i < j \leq n} \left( \frac{x_i}{x_j} ; q \right)_{a_i} \left( \frac{x_j}{x_i} q ; q \right)_{a_j} = \frac{(q ; q)_{a_1 + a_2 + \cdots + a_n}}{(q ; q)_{a_1} (q ; q)_{a_2} \cdots (q ; q)_{a_n}}.$$

Basic idea goes back to Ira Gessel’s proof of the Vandermonde determinant formula:

$$\prod_{1 \leq i < j \leq n} (x_i - x_j) = \sum_{\sigma \in S_n} (-1)^{\text{sgn} \sigma} \prod_{i=1}^{n} x_i^{\sigma(i)}.$$

Interpret LHS as sum over tournaments and find a sign-reversing involution on non-transitive tournaments.
1982, Doron published proof of Dyson’s conjecture based on Ira’s idea, *Discrete Math*:

\[
\prod_{1 \leq i, j \leq n} \left(1 - \frac{x_i}{x_j}\right)^{a_i} = \begin{pmatrix}
    a_1 + \cdots + a_n \\
    a_1, \ldots, a_n
\end{pmatrix}.
\]

LHS generates multi-tournaments, \(i\) and \(j\) play \(a_i + a_j\) games, \(i\) wins a total of \((n - 1) a_i\) games, sign is determined by parity of number of upsets (\(i < j\) but \(j\) beats \(i\)). RHS counts multi-words in \(a_1 1\)'s, \(a_2 2\)'s, \ldots
\[
\text{C.T.}\prod_{1 \leq i < j \leq n} \left( \frac{x_i}{x_j}; q \right)_{a_i} \left( \frac{x_j}{x_i}; q \right)_{a_j} = \frac{(q; q)_{a_1 + a_2 + \ldots + a_n}}{(q; q)_{a_1} (q; q)_{a_2} \ldots (q; q)_{a_n}}.
\]

Resulting words on RHS are weighted by Z-statistic. If \(W_{ij}\) is the subword in \(i\) and \(j\), then

\[
Z(W) = \sum_{i < j} \text{Maj}(W_{ij}),
\]

\(\text{Maj}(W_{ij}) = \text{sum of position of } j\text{'s that are followed by } i.\)
Resulting words on RHS are weighted by $Z$-statistic. If $W_{ij}$ is the subword in $i$ and $j$, then

$$Z(W) = \sum_{i<j} \text{Maj}(W_{ij}),$$

$\text{Maj}(W_{ij}) = \text{sum of position of } j\text{'s that are followed by } i.$

The gap lay in the proof that the $Z$-statistic is Mahonian, that the sum over all words $W$ in $a_1$ 1’s, $a_2$ 2’s, … of $q^{Z(W)}$ equals

$$\frac{(q;q)_{a_1+a_2+\cdots+a_n}}{(q;q)_{a_1} (q;q)_{a_2} \cdots (q;q)_{a_n}}.$$
Z-B: A Proof of Andrews’ $q$-Dyson Conjecture, *Discrete Math*, 1985

B. & Goulden: Constant Term Identities Extending the $q$-Dyson Theorem, *Transactions of the AMS*, 1985

———: The Generalized Plasma in One Dimension, *Communications in Mathematical Physics*, 1987

1980’s, Kathy O’Hara announced a somewhat complicated combinatorial proof that the Gaussian polynomial \((q; q)_{n+j}/(q; q)_n (q; q)_j\) is always unimodal. 


Doron showed that what Kathy actually proved is that

\[
\begin{bmatrix} n + j \\ j \end{bmatrix} = \sum q^{m_1^2 + \cdots + m_{j-1}^2} \prod_{i=1}^j \left[ (n + 2)i - M_{i-1} - M_{i+1} \right]/m_i - m_{i+1},
\]

\(m_1 \geq \cdots \geq m_j \geq m_{j+1} = 0, \quad m_0 = 0,\)

\(M_i = m_1 + \cdots + m_i, \quad M_j = j.\)

Unimodality follows by induction using the observation that each summand has the same mode.

Interpretation and generalization of the O’Hara-Zeilberger identity.

Benjamin, Quinn, Quinn, and Wójs, Composite fermions and integer partitions, *J. Combin. Th. A*, 2001
Alternating Sign Matrices

Kuperberg's representation

David Robbins (1942–2003)
$A_{n,k} =$ # of $n \times n$ alternating sign matrices with 1 in row 1, column $k$. 

\[
\begin{array}{cccccc}
1 & & & & & \\
1 & 1 & & & & \\
2 & 3 & 2 & & & \\
7 & 14 & 14 & 7 & & \\
42 & 105 & 135 & 105 & 42 & \\
429 & 1287 & 2002 & 2002 & 1287 & 429
\end{array}
\]
Conjecture:

\[
\frac{A_{n,k}}{A_{n,k+1}} = \frac{\binom{n-2}{k-1} + \binom{n-1}{k-1}}{\binom{n-2}{n-k-1} + \binom{n-1}{n-k-1}}
\]
The conjecture about the ratios implies that

\[ A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = \frac{1! \cdot 4! \cdot 7! \cdots (3n-2)!}{n! \cdot (n+1)! \cdots (2n-1)!} \]

But this was George Andrews’ formula for the number of descending plane partitions that fit into an \( n \times n \times n \) box.

George’s work was inspired by Ian Macdonald’s conjectured generating function for cyclically symmetric plane partitions.

“If I had to single out the most interesting open problem in all of enumerative combinatorics, this would be it.”

In the attempt to find a connection between descending plane partitions and alternating sign matrices that would prove their conjectures, Mills, Robbins, and Rumsey instead discovered a proof of Macdonald’s conjecture.

MRR, Proof of the Macdonald Conjecture, *Inv. Math.*, 1982
Oberwolfach, 1982:

“Dave’s first talk, about the MRR proof of Macdonald’s CSPP conjecture, was so good, and it hinted at the intriguing ASMs, that Dominique [Foata], (and everyone else!) begged Dave to give a second fifty-minute talk, about ASMs and their conjectured enumeration…

“On the way back, I was fortunate to share a train cabin with Dave, and I asked him lots of questions, and thus started my love-hate relationship with the ASM conjecture.”

Doron, Dave Robbins’ art of guessing, *Advances in Applied Mathematics*, 2005
1992: George Andrews proves Robbins conjecture that the number of totally symmetric, self-complementary plane partitions in an $n \times n \times n$ box is given by

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = \frac{1! \cdot 4! \cdot 7! \cdots (3n-2)!}{n! \cdot (n+1)! \cdots (2n-1)!}$$


Based on Rodney Baxter’s triangle-to-triangle relationship, a basic tool of statistical mechanics.

Published jointly by MAA and Cambridge University Press, 1999

Concludes with Doron’s brilliant proof of Robbins’ original conjecture:

\[
\frac{A_{n,k}}{A_{n,k+1}} = \binom{n-2}{k-1} + \binom{n-1}{k-1}
\]

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