Historical Reflections on the Fundamental Theorem Of (Integral) Calculus

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“The task of the educator is to make the child’s spirit pass again where its forefathers have gone, moving rapidly through certain stages but suppressing none of them. In this regard, the history of science must be our guide.”

Henri Poincaré
The history of mathematics informs teaching by

• helping students understand the process of mathematical discover,

• explaining the motivation behind definitions and assumptions, and

• illuminating conceptual difficulties.
Series, continuity, differentiation
1800–1850

Integration, structure of the real numbers
1850–1910
Historical Reflections on Teaching the Fundamental Theorem of Integral Calculus

David M. Bressoud

Abstract. This article explores the history of the Fundamental Theorem of Integral Calculus, from its origins in the 17th century through its formalization in the 19th century to its presentation in 20th century textbooks, and draws conclusions about what this historical development tells us about how to teach this fundamental insight of calculus.

1. INTRODUCTION. Nothing is considered more basic to calculus than the Fundamental Theorem of Integral Calculus, which is commonly presented as follows:

Fundamental Theorem of Integral Calculus (FTIC). For any function \( f \) that is continuous on the interval \([a, b]\),

\[
\frac{d}{dx} \int_{a}^{x} f(t) \, dt = f(x), \quad \text{for } a < x < b,
\]

(1)

and, if \( F'(x) = f(x) \) for all \( x \) in \([a, b]\), then

\[
\int_{a}^{b} f(t) \, dt = F(b) - F(a).
\]

(2)
What is the Fundamental Theorem of (Integral) Calculus?

Why is it fundamental?

Possible answer: Integration and differentiation are inverse processes of each other.

Problem with this answer: For most students, the working definition of integration is the inverse process of differentiation.
For any function $f$ that is continuous on the interval $[a,b]$,

\[
\frac{d}{dx} \int_{a}^{x} f(t) \, dt = f(x),
\]

and, if $F'(x) = f(x)$ for all $x$ in $[a,b]$, then

\[
\int_{a}^{b} f(t) \, dt = F(b) - F(a).
\]

Differentiation undoes integration

Definite integral provides an antiderivative

Integration undoes differentiation

An antiderivative provides a means of evaluating a definite integral.
The key to understanding this theorem is to know and appreciate that the definite integral is a limit of Riemann sums:

\[
\int_a^b f(x) \, dx \text{ is defined to be a limit over all partitions of } [a,b],
\]

\[
P = \left\{ a = x_0 < x_1 < \cdots < x_n = b \right\},
\]

\[
\int_a^b f(x) \, dx = \lim_{|P| \to 0} \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}),
\]

where \(|P| = \max_i (x_i - x_{i-1})\) and \(x_i^* \in [x_{i-1}, x_i] \).
Riemann’s habilitation of 1854:

Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe

\[
\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*) \Delta x_i
\]

Purpose of Riemann integral:

1. To investigate how discontinuous a function can be and still be integrable. Can be discontinuous on a dense set of points.

2. To investigate when an unbounded function can still be integrable. Introduce improper integral.
Riemann’s function: \( f(x) = \sum_{n=1}^{\infty} \frac{\{nx\}}{n^2} \)

\( \{x\} = \begin{cases} 
  x - (\text{nearest integer}), & \text{when this is } < \frac{1}{2}, \\
  0, & \text{when distance to nearest integer is } \frac{1}{2}.
\end{cases} \)

At \( x = \frac{a}{2b} \), \( \gcd(a,2b) = 1 \), the value of the function jumps by \( \frac{\pi^2}{8b^2} \).

\[ \frac{d}{dx} \int_a^x f(x) \, dx \neq f(x). \]
Riemann’s definition of the definite integral is not widely adopted by mathematicians until the 1870s.

First appearance of the *Fundamentalsatz der Integralrechnung* (*Fundamental Theorem of Integral Calculus*) in its modern form was in an appendix to a paper on Fourier series by Paul du Bois-Reymond in 1876.
Vito Volterra, 1881, found a bounded function $f$ with an anti-derivative $F$ so that $F'(x) = f(x)$ for all $x$, but there is *no* interval over which the definite integral of $f(x)$ exists.

\[ \int_a^b f(x) \, dx \neq F(b) - F(a). \]
Volterra’s Function

Two ingredients:

\[ F(x) = \begin{cases} 
  x^2 \sin\left(x^{-1}\right), & x \neq 0, \\
  0, & x = 0. 
\end{cases} \]

\[ F'(x) = \begin{cases} 
  2x \sin\left(x^{-1}\right) - \cos\left(x^{-1}\right), & x \neq 0, \\
  0, & x = 0. 
\end{cases} \]

A closed, nowhere dense subset of \([0,1]\) of measure \(\frac{1}{2}\).

*Remove middle \(\frac{1}{4}\), then two intervals of length \(1/16\),
four intervals of length \(1/64\), ...*
“…explanations drawn from algebraic technique … cannot be considered, in my opinion, except as heuristics that will sometimes suggest the truth, but which accord little with the accuracy that is so praised in the mathematical sciences.”
Niels Henrik Abel (1826):

“Cauchy is crazy, and there is no way of getting along with him, even though right now he is the only one who knows how mathematics should be done. What he is doing is excellent, but very confusing.”
Before Cauchy, the definite integral was defined as the difference of the values of an antiderivative at the endpoints.

This definition worked because all functions were assumed to be analytic, and thus an antiderivative always could be expressed in terms of a power series.
Cauchy, 1823, first explicit definition of definite integral as limit of sum of products

$$\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_{i-1})(x_{i} - x_{i-1}).$$

Purpose is to show that the definite integral is well-defined for any continuous function.
He now needs to connect this to the antiderivative. Using the mean value theorem for integrals, he proves that

\[
\frac{d}{dt} \int_{0}^{t} f(x) \, dx = f(t).
\]

By the mean value theorem, any function whose derivative is 0 must be constant. Therefore, any two functions with the same derivative differ by a constant. Therefore, if \( F \) is any antiderivative of \( f \), then

\[
\int_{0}^{t} f(x) \, dx = F(t) + C \quad \Rightarrow \quad \int_{a}^{b} f(x) \, dx = F(b) - F(a).
\]
The Fundamental Theorem of Integral Calculus was first stated in 1876.

It was first proven in 1823.

When was it first discovered?
Gottfried Leibniz: I shall now show that the general problem of quadratures [areas] can be reduced to the finding of a line that has a given law of tangency (declivitas), that is, for which the sides of the characteristic triangle have a given mutual relation. Then I shall show how this line can be described by a motion that I have invented.

Supplementum geometriae dimensoriae, Acta Eruditorum, 1693
Isaac Barrow: 1630–1677

“I add one or two theorems, which it will be seen are of great generality, and not lightly to be passed over.”

*Geometrical Lectures*, 1670

DF proportional to area VZGED, FT tangent to VIF at F, then DT is proportional to DF/DE.
James Gregory:  

*Geometriae pars universalis (The Universal Part of Geometry),* 1668

Given the equation of a curve, shows how to use integration to compute arc length.

Given the expression for arc length, shows how to find the equation of the corresponding curve. This requires proving that if $u$ represents the area under the curve whose height is $z$, then $z$ is the rate at which $u$ is changing.
Isaac Newton, the October 1666 Tract on Fluxions (unpublished):

“Problem 5: To find the nature of the crooked line [curve] whose area is expressed by any given equation.”

Let $y$ be the area under the curve $ac$, then “the motion by which $y$ increaseth will bee $bc = q$.”

“Problem 7: The nature of any crooked line being given to find its area, when it may bee.”
Isaac Newton, the October 1666 Tract on Fluxions (unpublished):

The area under the curve

\[ q = \frac{ax}{\sqrt{a^2 - x^2}} \]

is \(-a\sqrt{a^2 - x^2}\).

The area under the curve

\[ q = \sqrt{\frac{x^3}{a}} - \frac{e^2b}{x\sqrt{ax - x^2}} \]

is \(\frac{2}{5} \sqrt{\frac{x^5}{a}} + \frac{2e^2b}{ax} \sqrt{ax - x^2}\).
Nicole Oresme

*Tractatus de configurationibus qualitatum et motuum*

(*Treatise on the Configuration of Qualities and Motions*)

*Circa 1350*

Geometric demonstration that, under uniform acceleration, the distance traveled is equal to the distance traveled at constant average velocity.
Isaac Beeckman’s journal, 1618, unpublished in his lifetime

Beeckman gives a justification, using a limit argument, that if the ordinate of a curve represents velocity, then the accumulated area under that curve represents the distance traveled.
G.H. Hardy
*A Course of Pure Mathematics*, 1908:

“The ordinate of the curve is the derivative of the area, and the area is the integral of the ordinate.”
Three Lessons:

1. The real point of the FTIC is that there are two conceptually different but generally equivalent ways of interpreting integration: as antidifferentiation and as a limit of approximating sums.

2. The modern statement of the FTIC is the result of centuries of refinement of the original understanding and requires considerable unpacking if students are to understand and appreciate it.

3. The FTIC arose from a dynamical understanding of total change as an accumulation of small changes proportional to the instantaneous rate of change. This is where we need to begin to develop student understanding of the FTIC.

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