Proofs & Confirmations
The story of the alternating sign matrix conjecture

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These slides are available at www.macalester.edu/~bressoud/talks
IDA-CCR

Bill Mills

Howard Rumsey

David Robbins (1942–2003)

MAA Robbins Prize in algebra, combinatorics, or discrete math
Charles L. Dodgson

*aka* Lewis Carroll

“Condensation of Determinants,”
*Proceedings of the Royal Society, London*
1866
Square matrix:

- Entries are 0, 1, –1
- Row and column sums are +1
- Non-zero entries alternate in sign in each row
\[
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 1 & -1 & 0 & 1 \\
1 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
\]

<table>
<thead>
<tr>
<th>n</th>
<th>( A_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>42</td>
</tr>
<tr>
<td>5</td>
<td>429</td>
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<tr>
<td>6</td>
<td>7436</td>
</tr>
<tr>
<td>7</td>
<td>218348</td>
</tr>
<tr>
<td>8</td>
<td>10850216</td>
</tr>
<tr>
<td>9</td>
<td>911835460</td>
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</tbody>
</table>
How many $n \times n$ alternating sign matrices?

<table>
<thead>
<tr>
<th>$n$</th>
<th>$A_n$</th>
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<tbody>
<tr>
<td>1</td>
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<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>42 $\times 2 \times 3 \times 7$</td>
</tr>
<tr>
<td>5</td>
<td>429 $\times 3 \times 11 \times 13$</td>
</tr>
<tr>
<td>6</td>
<td>7436 $\times 2^2 \times 11 \times 13^2$</td>
</tr>
<tr>
<td>7</td>
<td>218348 $\times 2^2 \times 13^2 \times 17 \times 19$</td>
</tr>
<tr>
<td>8</td>
<td>10850216 $\times 2^3 \times 13 \times 17^2 \times 19^2$</td>
</tr>
<tr>
<td>9</td>
<td>911835460 $\times 2^2 \times 5 \times 17^2 \times 19^3 \times 23$</td>
</tr>
</tbody>
</table>

**Very Suspicious**
There is exactly one 1 in the first row.

\[
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 1 & -1 & 0 & 1 \\
1 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
\]

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<td>911835460</td>
</tr>
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There is exactly one 1 in the first row

\[
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 1 & -1 & 0 & 1 \\
1 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
\]

<table>
<thead>
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</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1+1</td>
</tr>
<tr>
<td>3</td>
<td>2+3+2</td>
</tr>
<tr>
<td>4</td>
<td>7+14+14+7</td>
</tr>
<tr>
<td>5</td>
<td>42+105+...</td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
</tr>
</tbody>
</table>
1

1   1

2   3   2

7   14  14   7

42  105 135 105 42

429 1287 2002 2002 1287 429
\[
\begin{array}{c}
1 \\
1 & 2/2 & 1 \\
2 & 2/3 & 3 & 3/2 & 2 \\
7 & 2/4 & 14 & 14 & 4/2 & 7 \\
42 & 2/5 & 105 & 135 & 105 & 5/2 & 42 \\
\end{array}
\]
Numerators:

1+1

1+1  1+2

1+1  2+3  1+3

1+1  3+4  3+6  1+4

1+1  4+5  6+10  4+10  1+5
Conjecture 1: \[ \frac{A_{n,k}}{A_{n,k+1}} = \frac{\binom{n-2}{k-1} + \binom{n-1}{k-1}}{\binom{n-2}{n-k-1} + \binom{n-1}{n-k-1}} \]
Conjecture 1: \[
\frac{A_{n,k}}{A_{n,k+1}} = \frac{\binom{n-2}{k-1} + \binom{n-1}{k-1}}{\binom{n-2}{n-k-1} + \binom{n-1}{n-k-1}}
\]

Conjecture 2 (corollary of Conjecture 1):

\[
A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = \frac{1! \cdot 4! \cdot 7! \cdots (3n-2)!}{n! \cdot (n+1)! \cdots (2n-1)!}
\]
Richard Stanley
Richard Stanley

Andrews’ Theorem: the number of descending plane partitions of size $n$ is

$$A_n = \prod_{j=0}^{n-1} \frac{(3j + 1)!}{(n + j)!} = \frac{1! \cdot 4! \cdot 7! \cdots (3n-2)!}{n! \cdot (n+1)! \cdots (2n-1)!}$$

George Andrews
What *is* a descending plane partition?
Percy A. MacMahon

Plane Partition

Work begun in 1897
Plane partition of 75

# of pp's of 75 = pp(75)

6 5 5 4 3 3
6 4 3 3 1
6 4 3 1 1
4 2 2 1
3 1 1
1 1 1
MacMahon finds a simple recursive algorithm for computing these numbers (based on generating function).

# of pp’s of 75 = pp(75) = 37,745,732,428,153
Symmetric Plane Partition

MacMahon *conjectures* a simple recursive algorithm for computing these numbers (based on generating function).
1971 Basil Gordon proves case for $n = \infty$

1977 George Andrews and Ian Macdonald independently prove general case
Cyclically Symmetric Plane Partition
**1979:** Ian Macdonald conjectures the generating function for cyclically symmetric plane partitions, producing an efficient recursive algorithm for computing these numbers.

“If I had to single out the most interesting open problem in all of enumerative combinatorics, this would be it.” Richard Stanley, review of *Symmetric Functions and Hall Polynomials*, *Bulletin of the AMS*, March, 1981.
1979, Andrews counts the number of cyclically symmetric plane partitions in an $n\times n\times n$ box.
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$L_1 = W_1 > L_2 = W_2 > L_3 = W_3 > \ldots$
1979, Andrews counts *descending* plane partitions

\[ L_1 > W_1 \geq L_2 > W_2 \geq L_3 > W_3 \geq \ldots \]

![Diagram of descending plane partitions](image)
Mills, Robbins, Rumsey Conjecture: # of $n \times n$ ASM’s with 1 at top of column $j$ equals # of DPP’s $\leq n$ with exactly $j-1$ parts of size $n$. 

\[
\begin{array}{cccccc}
6 & 6 & 6 & 4 & 3 \\
3 & 3 \\
2
\end{array}
\]
Discovered an *easier* proof of Andrews’ formula, using induction on $j$ and $n$.

Used this inductive argument to prove Macdonald’s conjecture


But they still didn’t have a proof of *their* conjecture!
Totally Symmetric Self-Complementary Plane Partitions

1983

David Robbins

Vertical flip of ASM = complement of DPP ?
Totally Symmetric Self-Complementary Plane Partitions
Robbins’ Conjecture: The number of TSSCPP’s in a $2n \times 2n \times 2n$ box is

$$\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = \frac{1! \cdot 4! \cdot 7! \cdots (3n-2)!}{n!(n+1)! \cdots (2n-1)!}.$$
Robbins’ Conjecture: The number of TSSCPP’s in a $2n \times 2n \times 2n$ box is

$$\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = \frac{1! \cdot 4! \cdot 7! \cdots (3n-2)!}{n! \cdot (n+1)! \cdots (2n-1)!}$$

1989: William Doran shows equivalent to counting lattice paths

1990: John Stembridge represents the counting function as a Pfaffian (built on insights of Gordon and Okada)

1992: George Andrews evaluates the Pfaffian, proves Robbins’ Conjecture
December, 1992
Doron Zeilberger announces a proof that # of ASM’s of size $n$ equals of TSSCPP’s in box of size $2n$. 
December, 1992

Doron Zeilberger announces a proof that # of ASM’s of size $n$ equals of TSSCPP’s in box of size $2n$.

Zeilberger’s proof is an 84-page tour de force, but it still left open the original conjecture:

\[
\frac{A_{n,k}}{A_{n,k+1}} = \frac{\binom{n-2}{k-1} + \binom{n-1}{k-1}}{\binom{n-2}{n-k-1} + \binom{n-1}{n-k-1}}
\]
1996 Kuperberg announces a simple proof

“Another proof of the alternating sign matrix conjecture,”
*International Mathematics Research Notices*

Greg Kuperberg
UC Davis
1996 Kuperberg announces a simple proof

“Another proof of the alternating sign matrix conjecture,”
*International Mathematics Research Notices*

Physicists had been studying ASM’s for decades, only they called them the *six-vertex model.*
\[
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 1 & -1 & 0 & 1 \\
1 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]

Horizontal = 1

Vertical = -1
1960’s
Rodney Baxter’s
Triangle-to-triangle relation

1980’s
Anatoli Izergin
Vladimir Korepin
1996

Doron Zeilberger uses this determinant to prove the original conjecture

“Proof of the refined alternating sign matrix conjecture,” *New York Journal of Mathematics*
The End

(which is really just the beginning)

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