The Truth of Proofs

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PowerPoint available at
www.macalester.edu/~bressoud/talks

Pacific Northwest Section
Juneau, AK
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Imre Lakatos, 1922–1974

Hungarian. Born Imre Lipschitz

Changed his name to Imre Lakatos (Locksmith) because he had shirts monogrammed IL.

1956: Hungarian uprising, flees to Vienna, then on to England

1961: PhD in Philosophy at Cambridge, with help from George Pólya.
Sir Karl Popper, *The Logic of Scientific Discovery*

Science advances by

1. Observing nature
2. Creating a theory to explain what is happening
3. Looking for consequences of this theory
4. Testing the predicted consequences
5. Adjusting the theory when predictions do not pan out

In science, nothing can be proven to be true. Real progress in science comes from establishing that something is false.
Lakatos: Is this relevant to Mathematics?
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Icosahedron
20 faces
30 edges
12 vertices

\[20 - 30 + 12 = 2\]

**Theorem (Euler):** For all polyhedra, \(V - E + F = 2\).

**Definition:** A *polyhedron* is a solid whose surface consists of polygonal faces.
$V = 16$
$E = 24$
$F = 12$

$16 - 24 + 12 = 4$

$V = 7$
$E = 12$
$F = 8$

$7 - 12 + 8 = 3$
For any polygon,  
\( V = E. \)

Can even be a self-intersecting polygon:
Small Stellated Dodecahedron

12 faces (pentagrams)
30 edges
12 vertices

\[12 - 30 + 12 = -6\]
Great Stellated Dodecahedron

\[ V = 20 \]
\[ E = 3 \times 20 \times 2 = 30 \]
\[ F = 2 \times 30 \times 5 = 12 \]
\[ 20 - 30 + 12 = 2 \]

What’s different?

Every closed curve made up of edges is the boundary of a chain of contiguous faces (faces that share an edge).
Here we have a closed circuit of faces that is not the boundary of the solid.

Here we have a closed circuit of edges that is not the boundary of a chain of contiguous faces.
The appendix from Lakatos’s *Proof and Refutations* would be the inspiration for my own *A Radical Approach to Real Analysis*
Cauchy, *Cours d’analyse*, 1821

“…explanations drawn from algebraic technique … cannot be considered, in my opinion, except as heuristics that will sometimes suggest the truth, but which accord little with the accuracy that is so praised in the mathematical sciences.”
“Cauchy is crazy, and there is no way of getting along with him, even though right now he is the only one who knows how mathematics should be done. What he is doing is excellent, but very confusing.”
Theorem 1. When the terms of a series are functions of a single variable $x$ and are continuous with respect to this variable in the neighborhood of a particular value where the series converges, the sum $S(x)$ of the series is also, in the neighborhood of this particular value, a continuous function of $x$.

$$S(x) = \sum_{k=1}^{\infty} f_k(x), \quad f_k \text{ continuous} \Rightarrow S \text{ continuous}$$
\[ S_n(x) = \sum_{k=1}^{n} f_k(x), \quad R_n(x) = S(x) - S_n(x) \]

Convergence \( \Rightarrow \) can make \( R_n(x) \) as small as we wish by taking \( n \) sufficiently large. \( S_n \) is continuous for \( n < \infty \).
\[ S_n(x) = \sum_{k=1}^{n} f_k(x), \quad R_n(x) = S(x) - S_n(x) \]

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\(S\) continuous at \(a\) if can force \(|S(x) - S(a)|\)

as small as we wish by restricting \(|x - a|\).
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\[ S \text{ continuous at } a \text{ if can force } |S(x) - S(a)| \]

as small as we wish by restricting \( |x - a| \).

\[
|S(x) - S(a)| = |S_n(x) + R_n(x) - S_n(a) - R_n(a)| \\
\leq |S_n(x) - S_n(a)| + |R_n(x)| + |R_n(a)|
\]
Abel, 1826:

“\(\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + \cdots\)
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\leq |S_n(x) - S_n(a)| + |R_n(x)| + |R_n(a)|
\]

\( x \) depends on \( n \) \quad \text{and} \quad \text{\( n \) depends on \( x \)}
$S_n(x) = \sum_{k=1}^{n} f_k(x), \quad R_n(x) = S(x) - S_n(x)$

Convergence $\Rightarrow$ can make $R_n(x)$ as small as we wish by taking $n$ sufficiently large. $S_n$ is continuous for $n < \infty$.

Uniform convergence eliminates the possibility that $n$ depends on $x$.

$$|S(x) - S(a)| = |S_n(x) + R_n(x) - S_n(a) - R_n(a)|$$
$$\leq |S_n(x) - S_n(a)| + |R_n(x)| + |R_n(a)|$$

$x$ depends on $n$ \quad $n$ depends on $x$
“A mathematician [is] an observer, a man who gazes at a distant range of mountains and notes down his observations … There are some peaks which he can distinguish easily, while others are less clear. He sees A sharply, while of B he can obtain only transitory glimpses. At last he makes out a ridge which leads from A, and following it to its end he discovers that it culminates in B. B is now fixed in his vision, and from this point he can proceed to further discoveries.”

G. H. Hardy, Rouse Ball Lecture, 1928.
I see the mathematician as someone who has parachuted into dense woods and needs to find her (or his) way back to familiar territory.
Proofs and Confirmations: The Story of the Alternating Sign Matrix Conjecture
Alternating sign matrix:

- Square matrix of 1’s, −1’s, and 0’s
- Each row and column adds to 1
- Nonzero entries in any row or column alternate in sign

\[
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 1 & -1 & 0 & 1 \\
1 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]
| $n$ | $A_n$ | \begin{align*}
&\text{How many } n \times n \text{ alternating sign matrices?}
\end{align*} |
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<tr>
<td>4</td>
<td>42</td>
<td>$= 2 \times 3 \times 7$</td>
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<td>5</td>
<td>429</td>
<td>$= 3 \times 11 \times 13$</td>
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<tr>
<td>6</td>
<td>7436</td>
<td>$= 2^2 \times 11 \times 13^2$</td>
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<tr>
<td>7</td>
<td>218348</td>
<td>$= 2^2 \times 13^2 \times 17 \times 19$</td>
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<tr>
<td>8</td>
<td>10850216</td>
<td>$= 2^3 \times 13 \times 17^2 \times 19^2$</td>
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<tr>
<td>9</td>
<td>911835460</td>
<td>$= 2^2 \times 5 \times 17^2 \times 19^3 \times 23$</td>
</tr>
</tbody>
</table>
There is exactly one 1 in the first row

\[
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 1 & -1 & 0 & 1 \\
1 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
\]

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\[
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 1 & -1 & 0 & 1 \\
1 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]

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<td>2+3+2</td>
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<td>4</td>
<td>7+14+14+7</td>
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<tr>
<td>5</td>
<td>42+105+…</td>
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Numerators:

1+1

1+1  1+2

1+1  2+3  1+3

1+1  3+4  3+6  1+4

1+1  4+5  6+10  4+10  1+5
Conjecture 1: \[
\frac{A_{n,k}}{A_{n,k+1}} = \frac{\binom{n-2}{k-1} + \binom{n-1}{k-1}}{\binom{n-2}{n-k-1} + \binom{n-1}{n-k-1}}
\]

Numerators:

\[
\begin{align*}
1+1 & \\
1+1 & 1+2 \\
1+1 & 2+3 & 1+3 \\
1+1 & 3+4 & 3+6 & 1+4 \\
1+1 & 4+5 & 6+10 & 4+10 & 1+5
\end{align*}
\]
Conjecture 1:  \[
\frac{A_{n,k}}{A_{n,k+1}} = \frac{\binom{n-2}{k-1} + \binom{n-1}{k-1}}{\binom{n-2}{n-k-1} + \binom{n-1}{n-k-1}}
\]

Conjecture 2 (corollary of Conjecture 1):
\[
A_n = \prod_{j=0}^{n-1} \frac{(3j + 1)!}{(n+j)!} = \frac{1! \cdot 4! \cdot 7! \cdots (3n-2)!}{n! \cdot (n+1)! \cdots (2n-1)!}
\]
Exactly the formula found by George Andrews for counting descending plane partitions.

Conjecture 2 (corollary of Conjecture 1):

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = \frac{1! \cdot 4! \cdot 7! \cdots (3n-2)!}{n! \cdot (n+1)! \cdots (2n-1)!}$$
Connections to partitions, determinant evaluations, orthogonal polynomials, counting lattice paths, tiling problems.


Previously described by Richard Stanley as “the most interesting open problem in all of enumerative combinatorics.”
1996 Kuperberg announces a simple proof of

\[ A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} \]

Physicists had been studying ASM’s for decades, only they call them the **six-vertex model**

Greg Kuperberg
UC Davis
\[
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 1 & -1 & 0 & 1 \\
1 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]

Horizontal = 1

Vertical = -1
Anatoli Izergin

Vladimir Korepin, SUNY Stony Brook

Rodney J. Baxter
Australian National University
1996

Doron Zeilberger uses the connection to statistical mechanics to prove the original conjecture

These slides are available at www.macalester.edu/~bressoud/talks

These books can be ordered at the MAA book exhibit.