Stories from the Development Of Real Analysis

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PowerPoint available at
www.macalester.edu/~bressoud/talks
“The task of the educator is to make the child’s spirit pass again where its forefathers have gone, moving rapidly through certain stages but suppressing none of them. In this regard, the history of science must be our guide.”

Henri Poincaré
Joseph Fourier

Niels Henrik Abel

Augustin-Louis Cauchy

Gustav Lejeune Dirichlet
1812

Napoleonic Europe
Joseph Fourier
44 years old
Prefect of the Department of Isère
Grenoble
Augustin-Louis Cauchy, 23 French military officer on medical leave in Paris
Niels Henrik Abel
10 years old
Gjerstad, Norway
Gustav Lejeune Dirichlet
7 years old
Düren, what is today Germany
Born 1768

Auxerre

Joseph Fourier

Grenoble

Paris

Versailles
École Normale Supérieure
Palais Bourbon
first home of the École Polytechnique
The Battle of the Pyramids
1798
Description de l’Égypte

151 learned men, including:

21 mathematicians
3 astronomers
17 civil engineers
13 naturalists and mining engineers
geographers
3 gunpowder engineers
4 architects
8 artists
10 mechanical artists
1 sculptor
15 interpreters
10 men of letters
22 printers

Eventually published in 37 volumes which appeared over the period 1809–1826

Fourier worked on the earliest volumes
The old Parliament Building, Grenoble
A stationary solution must satisfy Laplace’s equation:

\[ \frac{\partial^2 z}{\partial w^2} + \frac{\partial^2 z}{\partial x^2} = 0 \]
The problem:

The initial distribution of heat needs to be expressed as a linear combination of \( \cos(n\pi x/2) \), \(-1 < x < 1\), where \( n \) is an odd integer.

How do we express an initial distribution that is constant?
Fourier’s solution:

Express the constant 1 as

\[ 1 = \frac{2}{\pi} \left( \cos \left( \frac{\pi x}{2} \right) - \frac{1}{3} \cos \left( \frac{3\pi x}{2} \right) + \frac{1}{5} \cos \left( \frac{5\pi x}{2} \right) - \frac{1}{3} \cos \left( \frac{3\pi x}{2} \right) + \ldots \right) \]

Problem: This function of \( x \) satisfies

\[ f(x + 2) = -f(x) \]

This function is discontinuous at odd integers.
1811, awarded prize for the best paper on the propagation of heat, but:

... the manner in which the author arrives at these equations is not exempt of difficulties ... his analysis to integrate them still leaves something to be desired on the score of generality and even rigour.
Paris to Elba
1814
Elba to Paris
1815
1817: elected to French Academy of Sciences
Became Secretary of the Academy

1822: The Academy publishes *Analytic Theory of Heat*

Collège de France, home of the Academy of Sciences
Augustin-Louis Cauchy

Born 1789, Sceaux
(just south of Paris)

Laplace

Lagrange
Entered École Polytechnique in 1805
graduated 1807
entered École des Ponts and Chausées
(bridges and roads)
graduated 1809

1810, posted to Cherbourg to prepare the harbor for the invasion of England
Cauchy 1812

“Memoir on functions whose values are equal but of opposite sign when two of their variables are interchanged”

84-page tour de force

\[
\begin{vmatrix}
  x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \\
  \vdots & \vdots & \ddots & \vdots \\
  x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \\
  1 & 1 & \cdots & 1 \\
\end{vmatrix}
= \sum_{\sigma \in S_n} (-1)^{I(\sigma)} \prod_{i=1}^{n} x_i^{n-\sigma(i)}
\]

This function is 0 when \( x_i = x_j \) so it is divisible by \( \prod_{i<j} (x_i - x_j) \)

But both polynomials have same degree, so ratio is constant, \( = 1 \).
Unable to find academic employment in Paris, Cauchy worked on the Ourcq Canal until 1815.

1815: École Polytechnique
1816: elected member of the French Academy of Sciences
Niels Henrik Abel 1802–1829
Bernt Michael Helmboë
1795–1850

1817, became Abel’s mathematics teacher

1820, he wrote about Abel: “With the most incredible genius he unites ardour for and interest in mathematics such that he quite probably, if he lives, shall become one of the great mathematicians.”
Mémoire

sur

les équations algébriques

ou on démontre l'impossibilité de la résolution de l'équation générale
du cinquième degré

par

N. H. Abel.

Christiansia,
De l'imprimerie de Groendahl,
1824.
1825–26

Copenhagen
Berlin
Paris
Berlin
“My eyes have been opened in the most surprising manner. If you disregard the very simplest cases, there is in all of mathematics not a single infinite series whose sum had been rigorously determined. In other words, the most important parts of mathematics stand without foundation. It is true that most of it is valid, but that is very surprising. I struggle to find a reason for it, an exceedingly interesting problem.”

Letter to Holmboë from Berlin
“I am so anxious to hear your news! [...] This, the busiest capital on the continent, now feels to me like a desert. I know almost no one. [...] Up until now, I have only met Mr. Legendre, Cauchy, and Hachette, and several less famous but very capable mathematicians, Mr. Saigey, editor of the Bulletin of the Sciences, and Mr. Lejeune-Dirichlet, a Prussian who came to see me the other day believing I was a compatriot. He’s a mathematician of penetrating ability.”

1826 letter to Bernt Michael Holmboë
Lodged with Général Maximilien Foy 1775–1825

Mentored by Joseph Fourier among many others
Fermat’s Last Theorem

\[ x^5 + y^5 = z^5 \]

If there is a solution in positive, relatively prime integers, then exactly one of them is divisible by 5.

1825: Dirichlet proves that if there is a solution, the integer divisible by 5 cannot be even.

Legendre referees this paper and discovers a proof that the integer divisible by 5 cannot be odd.
“…explanations drawn from algebraic technique … cannot be considered, in my opinion, except as heuristics that will sometimes suggest the truth, but which accord little with the accuracy that is so praised in the mathematical sciences.”
From Abel’s 1826 letter to Holmboë:

“Cauchy is crazy, and there is no way of getting along with him, even though right now he is the only one who knows how mathematics should be done. What he is doing is excellent, but very confusing.”
Cauchy, Cours d’analyse, 1821, p. 120

**Theorem 1.** When the terms of a series are functions of a single variable $x$ and are continuous with respect to this variable in the neighborhood of a particular value where the series converges, the sum $S(x)$ of the series is also, in the neighborhood of this particular value, a continuous function of $x$.

$$S(x) = \sum_{k=1}^{\infty} f_k(x), \quad f_k \text{ continuous } \Rightarrow S \text{ continuous}$$
\[ S_n(x) = \sum_{k=1}^{n} f_k(x), \quad R_n(x) = S(x) - S_n(x) \]

Convergence \( \Rightarrow \) can make \( R_n(x) \) as small as we wish by taking \( n \) sufficiently large. \( S_n \) is continuous for \( n < \infty \).

\[ S \text{ continuous at } a \text{ if can force } |S(x) - S(a)| \]

as small as we wish by restricting \(|x - a|\).

\[
|S(x) - S(a)| = |S_n(x) + R_n(x) - S_n(a) - R_n(a)| \\
\leq |S_n(x) - S_n(a)| + |R_n(x)| + |R_n(a)|
\]
Abel, 1826:

“It appears to me that this theorem suffers exceptions.”

\[
\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots
\]
\[ S_n(x) = \sum_{k=1}^{n} f_k(x), \quad R_n(x) = S(x) - S_n(x) \]

Convergence \(\Rightarrow\) can make \(R_n(x)\) as small as we wish by taking \(n\) sufficiently large. \(S_n\) is continuous for \(n < \infty\).

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|S(x) - S(a)| = |S_n(x) + R_n(x) - S_n(a) - R_n(a)|
\]
\[\leq |S_n(x) - S_n(a)| + |R_n(x)| + |R_n(a)|\]

\(x\) depends on \(n\) \quad \(n\) depends on \(x\)
Before Cauchy, the definite integral was defined as the difference of the values of an antiderivative at the endpoints.

This definition worked because all functions were assumed to be analytic, and thus an antiderivative always could be expressed in terms of a power series.
Cauchy, 1823, first explicit definition of definite integral as limit of sum of products

\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}).
\]

Purpose is to show that the definite integral is well-defined for \textit{any} continuous function.
He now needs to connect this to the antiderivative. Using the mean value theorem for integrals, he proves that

\[ \frac{d}{dt} \int_0^t f(x) \, dx = f(t). \]

By the mean value theorem, any function whose derivative is 0 must be constant. Therefore, any two functions with the same derivative differ by a constant. Therefore, if \( F \) is any antiderivative of \( f \), then

\[ \int_0^t f(x) \, dx = F(t) + C \quad \Rightarrow \quad \int_a^b f(x) \, dx = F(b) - F(a). \]
1827, Dirichlet takes a position at the University of Breslau (today’s Wrocław, Poland)

1828, moved to Berlin

1831, appointed to Berlin Academy
Sur la convergence des séries trigonométriques qui servent à représenter une fonction arbitraire entre des limites données.

(Par Mr. Lejeune-Dirichlet, prof. de mathém.)

MSC-Class : 51-03 01A16

Les séries de sinus et de cosinus, au moyen desquelles on peut représenter une fonction arbitraire dans un intervalle donné, jouissent entre autres propriétés remarquables aussi de celle d’être convergentes. Cette propriété n’avait pas échappée au géomètre illustre qui a ouvert une nouvelle carrière aux applications de l’analyse, en y introduisant la manière d’exprimer les fonctions arbitraires dont il est question; elle se trouve énoncée dans le Mémoire qui contient ses premières recherches sur la chaleur. Mais personne, que je sache, n’en a donné jusqu’à présent une démonstration générale. Je ne connais sur cet objet qu’un travail dû à M. CAUCHY et qui fait partie des Mémoires de l’Académie des sciences de Paris pour l’année 1823. L’auteur de ce travail avoue lui aussi avoir eu d’abord une idée de la définition d’une certaine fonction...
Niels Henrik Abel died of tuberculosis on April 6, 1829.

One year later, he was posthumously awarded the Grand Prix from the French Academy of Sciences (together with Evariste Galois) for his work on solutions by radicals to algebraic equations.
Through the 1820’s, Fourier mentored many young mathematicians including Gustav Dirichlet, Sophie Germain, Joseph Liouville, Claude Navier, Charles Sturm.

Loved to recount tales of his adventures in Egypt.

Fourier died on May 16, 1830.
Augustin-Louis Cauchy, a fervent royalist, left Paris immediately after the Revolution of 1830.

He returned in 1848, following the fall of Louis-Philippe.

He died in 1857.
Gustav Lejeune Dirichlet continued to thrive in Berlin.

Among his students was Bernhard Riemann.

In 1855, following the death of his friend Carl Friedrich Gauss, he succeeded to Gauss’s chair at the University of Göttingen.

Dirichlet died in 1859.