Proofs & Confirmations
The story of the
alternating sign matrix conjecture

David M. Bressoud
Macalester College

Park City, Utah
July 17, 2012

These slides are available at
www.macalester.edu/~bressoud/talks
IDA-CCR

Bill Mills

Howard Rumsey

David Robbins (1942–2003)

MAA Robbins Prize in algebra, combinatorics, or discrete math
Charles L. Dodgson
aka Lewis Carroll

“Condensation of Determinants,”
*Proceedings of the Royal Society, London*
1866
Square matrix:

- Entries are 0, 1, –1
- Row and column sums are +1
- Non-zero entries alternate in sign in each row
\[
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 1 & -1 & 0 & 1 \\
1 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]

<table>
<thead>
<tr>
<th>( n )</th>
<th>( A_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>42</td>
</tr>
<tr>
<td>5</td>
<td>429</td>
</tr>
<tr>
<td>6</td>
<td>7436</td>
</tr>
<tr>
<td>7</td>
<td>218348</td>
</tr>
<tr>
<td>8</td>
<td>10850216</td>
</tr>
<tr>
<td>9</td>
<td>911835460</td>
</tr>
<tr>
<td>$n$</td>
<td>$A_n$</td>
</tr>
<tr>
<td>-----</td>
<td>-------</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>42 $= 2 \times 3 \times 7$</td>
</tr>
<tr>
<td>5</td>
<td>429 $= 3 \times 11 \times 13$</td>
</tr>
<tr>
<td>6</td>
<td>7436 $= 2^2 \times 11 \times 13^2$</td>
</tr>
<tr>
<td>7</td>
<td>218348 $= 2^2 \times 13^2 \times 17 \times 19$</td>
</tr>
<tr>
<td>8</td>
<td>10850216 $= 2^3 \times 13 \times 17^2 \times 19^2$</td>
</tr>
<tr>
<td>9</td>
<td>911835460 $= 2^2 \times 5 \times 17^2 \times 19^3 \times 23$</td>
</tr>
</tbody>
</table>
There is exactly one 1 in the first row.

\[
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 1 & -1 & 0 & 1 \\
1 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]

<table>
<thead>
<tr>
<th>$n$</th>
<th>$A_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>42</td>
</tr>
<tr>
<td>5</td>
<td>429</td>
</tr>
<tr>
<td>6</td>
<td>7436</td>
</tr>
<tr>
<td>7</td>
<td>218348</td>
</tr>
<tr>
<td>8</td>
<td>10850216</td>
</tr>
<tr>
<td>9</td>
<td>911835460</td>
</tr>
</tbody>
</table>
There is exactly one 1 in the first row

\[
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 1 & -1 & 0 & 1 \\
1 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
\]

<table>
<thead>
<tr>
<th>n</th>
<th>$A_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1+1</td>
</tr>
<tr>
<td>3</td>
<td>2+3+2</td>
</tr>
<tr>
<td>4</td>
<td>7+14+14+7</td>
</tr>
<tr>
<td>5</td>
<td>42+105+...</td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
</tr>
</tbody>
</table>
\[
\begin{array}{ccccccc}
\text{429} & \text{1287} & \text{2002} & \text{2002} & \text{1287} & \text{429} \\
\text{42} & \text{105} & \text{135} & \text{105} & \text{42} & \\
\text{7} & + & \text{14} & + & \text{14} & + & \text{7} \\
\text{2} & \text{3} & \text{2} & \\
\text{1} & \text{1} & \text{1} & \\
\text{1} & \\
\end{array}
\]

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
Numerators:

1+1

1+1  1+2

1+1  2+3  1+3

1+1  3+4  3+6  1+4

1+1  4+5  6+10  4+10  1+5
Conjecture 1: \[
\frac{A_{n,k}}{A_{n,k+1}} = \frac{\binom{n-2}{k-1} + \binom{n-1}{k-1}}{\binom{n-2}{n-k-1} + \binom{n-1}{n-k-1}}
\]
Conjecture 1: \[ \frac{A_{n,k}}{A_{n,k+1}} = \frac{\binom{n-2}{k-1} + \binom{n-1}{k-1}}{\binom{n-2}{n-k-1} + \binom{n-1}{n-k-1}} \]

Conjecture 2 (corollary of Conjecture 1):

\[ A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = \frac{1! \cdot 4! \cdot 7! \cdots (3n-2)!}{n! \cdot (n+1)! \cdots (2n-1)!} \]
Andrews’ Theorem: the number of descending plane partitions of size $n$ is

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = \frac{1! \cdot 4! \cdot 7! \cdots (3n-2)!}{n! \cdot (n+1)! \cdots (2n-1)!}$$
What is a descending plane partition?
Percy A. MacMahon

Plane Partition

Work begun in 1897
Plane partition of 75

# of pp's of 75 = \( pp(75) \)
MacMahon finds a simple recursive algorithm for computing these numbers (based on generating function).

# of pp’s of 75 = pp(75) = 37,745,732,428,153
MacMahon conjectures a simple recursive algorithm for computing these numbers (based on generating function).
1971 Basil Gordon proves case for $n = \infty$

1977 George Andrews and Ian Macdonald independently prove general case
Cyclically Symmetric Plane Partition
1979: Ian Macdonald conjectures the generating function for cyclically symmetric plane partitions, producing an efficient recursive algorithm for computing these numbers.

“If I had to single out the most interesting open problem in all of enumerative combinatorics, this would be it.” Richard Stanley, review of *Symmetric Functions and Hall Polynomials*, *Bulletin of the AMS*, March, 1981.
1979, Andrews counts the number of cyclically symmetric plane partitions in an $n \times n \times n$ box.
1979, Andrews counts the number of cyclically symmetric plane partitions in an $n \times n \times n$ box.
1979, Andrews counts the number of cyclically symmetric plane partitions in an $n \times n \times n$ box.
1979, Andrews counts the number of cyclically symmetric plane partitions in an $n \times n \times n$ box.
1979, Andrews counts the number of cyclically symmetric plane partitions in an $n \times n \times n$ box.

$L_1 = W_1 > L_2 = W_2 > L_3 = W_3 > \ldots$
1979, Andrews counts **descending** plane partitions

\[ L_1 > W_1 \geq L_2 > W_2 \geq L_3 > W_3 \geq \ldots \]
Mills, Robbins, Rumsey Conjecture: \# of $n \times n$ ASM’s with 1 at top of column $j$ equals \# of DPP’s $\leq n$ with exactly $j-1$ parts of size $n$. 
Discovered an *easier* proof of Andrews’ formula, using induction on $j$ and $n$.

Used this inductive argument to prove Macdonald’s conjecture


But they still didn’t have a proof of *their* conjecture!
Totally Symmetric Self-Complementary Plane Partitions

1983

David Robbins

Vertical flip of ASM = complement of DPP?
Totally Symmetric Self-Complementary Plane Partitions
Robbins’ Conjecture: The number of TSSCPP’s in a $2n \times 2n \times 2n$ box is

$$\prod_{j=0}^{n-1} \frac{(3j + 1)!}{(n + j)!} = \frac{1! \cdot 4! \cdot 7! \cdots (3n - 2)!}{n! \cdot (n + 1)! \cdots (2n - 1)!}$$
Robbins’ Conjecture: The number of TSSCPP’s in a $2n \times 2n \times 2n$ box is

$$\prod_{j=0}^{n-1} \frac{(3j + 1)!}{(n + j)!} = \frac{1! \cdot 4! \cdot 7! \cdots (3n - 2)!}{n! (n + 1)! \cdots (2n-1)!}$$

1989: William Doran shows equivalent to counting lattice paths

1990: John Stembridge represents the counting function as a Pfaffian (built on insights of Gordon and Okada)

1992: George Andrews evaluates the Pfaffian, proves Robbins’ Conjecture
December, 1992

Doron Zeilberger announces a proof that
# of ASM’s of size $n$
equals of TSSCPP’s in
box of size $2n$. 
December, 1992

Doron Zeilberger announces a proof that 
# of ASM’s of size $n$ 
equals of TSSCPP’s in box of size $2n$.

Zeilberger’s proof is an 84-page tour de force, but it still left open the original conjecture:

\[
\frac{A_{n,k}}{A_{n,k+1}} = \frac{\binom{n-2}{k-1} + \binom{n-1}{k-1}}{\binom{n-2}{n-k-1} + \binom{n-1}{n-k-1}}
\]
1996 Kuperberg announces a simple proof

“Another proof of the alternating sign matrix conjecture,”
*International Mathematics Research Notices*

Greg Kuperberg
UC Davis
1996 Kuperberg announces a simple proof

“Another proof of the alternating sign matrix conjecture,”
*International Mathematics Research Notices*

Physicists had been studying ASM’s for decades, only they called them the *six-vertex model*.
\[
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 1 & -1 & 0 & 1 \\
1 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
\]

Horizontal = 1

Vertical = -1
1960’s
Rodney Baxter’s
Triangle-to-triangle relation

1980’s
Anatoli Izergin
Vladimir Korepin
1996
Doron Zeilberger uses this determinant to prove the original conjecture

The End

(which is really just the beginning)

These slides can be downloaded from
www.macalester.edu/~bressoud/talks