Stories from the Development of Real Analysis

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“The task of the educator is to make the child’s spirit pass again where its forefathers have gone, moving rapidly through certain stages but suppressing none of them. In this regard, the history of science must be our guide.”

Henri Poincaré
A Radical Approach to
Lebesgue’s Theory of Integration

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Series, continuity, differentiation
1800–1850

Integration, structure of the real numbers
1850–1910
1. Cauchy and uniform convergence

2. The Fundamental Theorem of Calculus
“What Weierstrass — Cantor — did was very good. That's the way it had to be done. But whether this corresponds to what is in the depths of our consciousness is a very different question …

Nikolai Luzin
1883–1950
… I cannot but see a stark contradiction between the intuitively clear fundamental formulas of the integral calculus and the incomparably artificial and complex work of the ‘justification’ and their ‘proofs’.

Nikolai Luzin
1883–1950
Cauchy, *Cours d’analyse*, 1821

“…explanations drawn from algebraic technique … cannot be considered, in my opinion, except as heuristics that will sometimes suggest the truth, but which accord little with the accuracy that is so praised in the mathematical sciences.”
\[ 1 = 1 - x + x - x^2 + x^2 - x^3 + \cdots \]
\[ = (1 - x) + (x - x^2) + (x^2 - x^3) + \cdots \]
\[ = (1 - x) + x(1 - x) + x^2 (1 - x) + \cdots \]
\[ = (1 - x)(1 + x + x^2 + \cdots) \]
\[ \frac{1}{1 - x} = 1 + x + x^2 + \cdots \]
\[
1 = 1 - x + x - x^2 + x^2 - x^3 + \cdots \\
= (1 - x) + (x - x^2) + (x^2 - x^3) + \cdots \\
= (1 - x) + x(1 - x) + x^2 (1 - x) + \cdots \\
= (1 - x) (1 + x + x^2 + \cdots ) \\
\frac{1}{1-x} = 1 + x + x^2 + \cdots \\
-1 = \frac{1}{1-2} = 1 + 2 + 2^2 + \cdots 
\]
\[
1 = 1 - x + x - x^2 + x^2 - x^3 + \cdots - x^N + x^N \\
= (1 - x) + (x - x^2) + (x^2 - x^3) + \cdots + (x^{N-1} - x^N) + x^N \\
= (1 - x) + x(1 - x) + x^2(1 - x) + \cdots + x^{N-1}(1 - x) + x^N \\
= (1 - x)(1 + x + x^2 + \cdots + x^{N-1}) + x^N \\
\frac{1}{1 - x} = 1 + x + x^2 + \cdots + x^{N-1} + \frac{x^N}{1 - x}
\]
Niels Henrik Abel (1826): 

“Cauchy is crazy, and there is no way of getting along with him, even though right now he is the only one who knows how mathematics should be done. What he is doing is excellent, but very confusing.”
Cauchy, Cours d’analyse, 1821, p. 120

Theorem 1. When the terms of a series are functions of a single variable $x$ and are continuous with respect to this variable in the neighborhood of a particular value where the series converges, the sum $S(x)$ of the series is also, in the neighborhood of this particular value, a continuous function of $x$.

$$S(x) = \sum_{k=1}^{\infty} f_k(x), \quad f_k \text{ continuous} \implies S \text{ continuous}$$
\[ S_n(x) = \sum_{k=1}^{n} f_k(x), \quad R_n(x) = S(x) - S_n(x) \]

Convergence \( \Rightarrow \) can make \( R_n(x) \) as small as we wish by taking \( n \) sufficiently large. \( S_n \) is continuous for \( n < \infty \).
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\(S\) continuous at \(a\) if can force \(|S(x) - S(a)|\)
as small as we wish by restricting \(|x - a|\).
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\[ S \text{ continuous at } a \text{ if can force } |S(x) - S(a)| \]

as small as we wish by restricting \( |x - a| \).

\[
|S(x) - S(a)| = |S_n(x) + R_n(x) - S_n(a) - R_n(a)| \\
\leq |S_n(x) - S_n(a)| + |R_n(x)| + |R_n(a)|
\]
Abel, 1826:

“It appears to me that this theorem suffers exceptions.”

$$\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots$$
\[ S_n(x) = \sum_{k=1}^{n} f_k(x), \quad R_n(x) = S(x) - S_n(x) \]

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\]

\( x \) depends on \( n \) \quad \( n \) depends on \( x \)
“If even Cauchy can make a mistake like this, how am I supposed to know what is correct?”
What is the Fundamental Theorem of Calculus?

Why is it fundamental?
The Fundamental Theorem of Calculus (evaluation part):

If \( F'(x) = f(x) \), then \( \int_{a}^{b} f(x) \, dx = F(b) - F(a) \).

Differentiate then Integrate = original fcn (up to constant)

The Fundamental Theorem of Calculus (antiderivative part):

If \( f \) is continuous, then \( \frac{d}{dt} \int_{0}^{t} f(x) \, dx = f(t) \).

Integrate then Differentiate = original fcn
The Fundamental Theorem of Calculus (evaluation part):

If \( F'(x) = f(x) \), then \( \int_a^b f(x) \, dx = F(b) - F(a) \).

I.e., integration and differentiation are inverse processes, but isn’t this the definition of integration?
Cauchy, 1823, first explicit definition of definite integral as limit of sum of products

\[ \int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}); \]

mentions the fact that

\[ \frac{d}{dt} \int_0^t f(x)dx = f(t) \]

en route to his definition of the indefinite integral.
Fundamental Theorem for Integrals

De L’Analyse Infinitésimal, Charles de Freycinet, 1860
Fundamental Theorem for Integrals

*De L’Analyse Infinitésimal*, Charles de Freycinet, 1860

1831–1889

Fundamental Theorem of Integral Calculus

used by Paul du Bois-Reymond in appendix to paper on trigonometric series, 1876.

1828–1923

G. H. Hardy, 2nd edition of *A Course of Pure Mathematics*, 1914, refers to it as the *Fundamental Theorem of Calculus*.
The real FTC:
There are two distinct ways of viewing integration:

• As a limit of a sum of products (Riemann sum),

• As the inverse process of differentiation.

The power of calculus comes precisely from their equivalence.
Riemann’s habilitation of 1854:

*Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe*

\[
\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f\left(x_{i}^{*}\right) \Delta x_i
\]

**Purpose of Riemann integral:**

1. To investigate how discontinuous a function can be and still be integrable. Can be discontinuous on a dense set of points.

2. To investigate when an unbounded function can still be integrable. Introduce improper integral.
Riemann’s function: \[ f(x) = \sum_{n=1}^{\infty} \frac{\{nx\}}{n^2} \]

\[ \{x\} = \begin{cases} 
  x - \text{(nearest integer)}, & \text{when this is } < \frac{1}{2}, \\
  0, & \text{when distance to nearest integer is } \frac{1}{2}
\end{cases} \]

At \( x = \frac{a}{2b}, \) \( \gcd(a, 2b) = 1, \) the function jumps by \( \frac{\pi^2}{8b^2} \)
The Fundamental Theorem of Calculus (antiderivative part):

If $f$ is continuous, then \[
\frac{d}{dt} \int_0^t f(x) \, dx = f(t).
\]

Integrate then Differentiate $\Rightarrow$ original function

This part of the FTC does not hold at points where $f$ is not continuous.
The Fundamental Theorem of Calculus (evaluation part):

If \( F'(x) = f(x) \), then \( \int_a^b f(x)dx = F(b) - F(a) \).

Differentiate then Integrate = original fcn (up to constant)

Volterra, 1881, constructed function with bounded derivative that is \textbf{not} Riemann integrable.
\[ F(x) = \begin{cases} 
  x^2 \sin\left(\frac{1}{x}\right), & x \neq 0, \\
  0, & x = 0. 
\end{cases} \]
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\[ F'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right), \quad x \neq 0. \]
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\[ F'(0) = \lim_{h \to 0} \frac{F(h) - F(0)}{h} \]

\[ = \lim_{h \to 0} \frac{h^2 \sin\left(\frac{1}{h}\right)}{h} \]

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\[ = \lim_{h \to 0} \frac{h^2 \sin\left(\frac{1}{h}\right)}{h} \]

\[ = \lim_{h \to 0} h \sin\left(\frac{1}{h}\right) = 0. \]

\[ \lim F'(x) \text{ does not exist, but } F'(0) \text{ does exist (and equals 0).} \]
Cantor’s Set

Remove \([1/3, 2/3]\)
Cantor’s Set

Remove $[1/3,2/3]$  

Remove $[1/9,2/9]$ and $[7/9,8/9]$
Cantor’s Set

Remove $[1/3, 2/3]$

Remove $[1/9, 2/9]$ and $[7/9, 8/9]$

Cantor’s Set

\[
1 - \left( \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \cdots \right) \\
= 1 - \frac{1}{3} \left( 1 + \left( \frac{2}{3} \right) + \left( \frac{2}{3} \right)^2 + \left( \frac{2}{3} \right)^3 + \cdots \right) \\
= 1 - \frac{1}{3} \cdot \frac{1}{1 - \frac{2}{3}} = 1 - 1 = 0 .
\]

What’s left has measure 0.
Create a new set like the Cantor set except

the first middle piece only has length 1/4

each of the next two middle pieces only have length 1/16

the next four pieces each have length 1/64, etc.

The amount left has size

\[
1 - \left( \frac{1}{4} + \frac{2}{16} + \frac{4}{64} + \frac{2^3}{4^4} + \cdots \right)
\]

\[
= 1 - \frac{1}{4} \left( 1 + \left( \frac{2}{4} \right) + \left( \frac{2}{4} \right)^2 + \left( \frac{2}{4} \right)^3 + \cdots \right)
\]

\[
= 1 - \frac{1}{4} \cdot \frac{1}{1 - \frac{1}{2}} = \frac{1}{2}
\]
Perfect set: equals its set of limit points

Nowhere dense: every interval contains subinterval with no points of the set

Positive measure: the missing points are contained in a union of intervals whose lengths add up to less than 1.

Henry Smith was the first person to show how to get such a set, 1875.
We’ll call this set SVC(4) (for Smith-Volterra-Cantor).

It has two important characteristics:

1. SVC(4) is **nowhere dense**. No matter how small an interval \((a, b) \subseteq [0,1]\) we take, there will be an entire subinterval of points, \((\alpha, \beta) \subseteq (a, b)\), that are not in SVC(4).

2. Given *any* collection of subintervals of \([0,1]\), whose union contains SVC(4), the sum of the lengths of these intervals is at least 1/2.
Volterra’s construction:

Start with the function \( F(x) = \begin{cases} x^2 \sin\left(\sqrt{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases} \)

Restrict to the interval \([0, 1/8]\), except find the largest value of \( x \) on this interval at which \( F'(x) = 0 \), and keep \( F \) constant from this value all the way to \( x = 1/8 \).
Volterra’s construction:

To the right of $x = 1/8$, take the mirror image of this function: for $1/8 < x < 1/4$, and outside of $[0,1/4]$, define this function to be 0. Call this function $f_i(x)$. 
Volterra’s construction:

\[ f_1(x) \] is a differentiable function for all values of \( x \), but

\[ \lim_{x \to 0^+} f_1'(x) \quad \text{and} \quad \lim_{x \to \frac{1}{4}^-} f_1'(x) \] do not exist.
Now we slide this function over so that the portion that is not identically 0 is in the interval \([3/8,5/8]\), that middle piece of length 1/4 taken out of SVC(4).
We follow the same procedure to create a new function, \( f_2(x) \), that occupies the interval \([0,1/16]\) and is 0 outside this interval, then slide it over so that it is centered at \( 3/16 = 0.1875 \).
We insert a copy of $f_2(x)$ into each interval of length $1/16$ that was removed from $SVC(4)$. 
We do this for every interval removed from SVC(4). In the limit as each interval gets its copy of two pieces of $x^2 \sin(1/x)$, we get a function that is continuous and differentiable at all values of $x$, and the derivative stays bounded.
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For the endpoints of the removed intervals, every neighborhood, no matter how small, contains points where the derivative is $+1$ and points where the derivative is $-1$. 
If we now take any partition of $[0,1]$, there will be subintervals of total length at least $1/2$ that contain endpoints of the removed intervals in their interior.
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The derivative of Volterra’s function is not integrable.
Lessons:

1. Riemann’s definition is not intuitively natural. Students think of integration as inverse of differentiation. Cauchy definition is easier to comprehend.
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1. Riemann’s definition is not intuitively natural. Students think of integration as inverse of differentiation. Cauchy definition is easier to comprehend.

2. Emphasize FTC as connecting two very different ways of interpreting integration. Go back to calling it the Fundamental Theorem of Integral Calculus.

3. Need to let students know that these interpretations of integration really are different.
This presentation is available at
www.macalester.edu/~bressoud/talks