Proofs & Confirmations
The story of the alternating sign matrix conjecture

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Macalester College

MAA Carriage House
September 19, 2007
Charles L. Dodgson
aka Lewis Carroll

“Condensation of Determinants,"
Proceedings of the Royal Society, London
1866
\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
\]

<table>
<thead>
<tr>
<th>( n )</th>
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<td>7</td>
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<td>4</td>
<td>$42 = 2 \times 3 \times 7$</td>
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<tr>
<td>5</td>
<td>$429 = 3 \times 11 \times 13$</td>
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<tr>
<td>6</td>
<td>$7436 = 2^2 \times 11 \times 13^2$</td>
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<tr>
<td>7</td>
<td>$218348 = 2^2 \times 13^2 \times 17 \times 19$</td>
</tr>
<tr>
<td>8</td>
<td>$10850216 = 2^3 \times 13 \times 17^2 \times 19^2$</td>
</tr>
<tr>
<td>9</td>
<td>$911835460 = 2^2 \times 5 \times 17^2 \times 19^3 \times 23$</td>
</tr>
</tbody>
</table>
There is exactly one 1 in the first row

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\end{pmatrix}
\]

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<tr>
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<td>911835460</td>
</tr>
</tbody>
</table>
There is exactly one 1 in the first row.

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
\]

<table>
<thead>
<tr>
<th>( n )</th>
<th>( A_n )</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>1</td>
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<tr>
<td>2</td>
<td>1+1</td>
</tr>
<tr>
<td>3</td>
<td>2+3+2</td>
</tr>
<tr>
<td>4</td>
<td>7+14+14+7</td>
</tr>
<tr>
<td>5</td>
<td>42+105+…</td>
</tr>
<tr>
<td>6</td>
<td></td>
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<tr>
<td>7</td>
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<td>8</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
</tr>
</tbody>
</table>
\begin{array}{cccccc}
1 \\
1 & 1 \\
2 & 3 & 2 \\
7 & 14 & 14 & 7 \\
42 & 105 & 135 & 105 & 42 \\
429 & 1287 & 2002 & 2002 & 1287 & 429 \\
\end{array}
<table>
<thead>
<tr>
<th></th>
<th>42</th>
<th>105</th>
<th>135</th>
<th>105</th>
<th>42</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>+</td>
<td>14</td>
<td>+</td>
<td>14</td>
<td>+</td>
</tr>
</tbody>
</table>

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2/2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2/3</td>
<td>3</td>
<td>3/2</td>
</tr>
<tr>
<td>7</td>
<td>2/4</td>
<td>14</td>
<td>5/5</td>
</tr>
<tr>
<td>42</td>
<td>2/5</td>
<td>105</td>
<td>7/9</td>
</tr>
<tr>
<td>429</td>
<td>2/6</td>
<td>1287</td>
<td>9/14</td>
</tr>
</tbody>
</table>
Numerators:

1+1
1+1   1+2
1+1   2+3   1+3
1+1   3+4   3+6   1+4
1+1   4+5   6+10  4+10   1+5
Numerators: \[1 + 1\]

\[1 + 1 \quad 1 + 2\]

\[1 + 1 \quad 2 + 3 \quad 1 + 3\]

\[1 + 1 \quad 3 + 4 \quad 3 + 6 \quad 1 + 4\]

\[1 + 1 \quad 4 + 5 \quad 6 + 10 \quad 4 + 10 \quad 1 + 5\]

Conjecture 1: \[
\frac{A_{n,k}}{A_{n,k+1}} = \frac{\binom{n-2}{k-1} + \binom{n-1}{k-1}}{\binom{n-2}{n-k-1} + \binom{n-1}{n-k-1}}
\]
Conjecture 1: \[ \frac{A_{n,k}}{A_{n,k+1}} = \frac{(n-2) \binom{k-1}{k-1} + (n-1) \binom{n-1}{k-1}}{\binom{n-2}{n-k-1} + \binom{n-1}{n-k-1}} \]

Conjecture 2 (corollary of Conjecture 1):
\[ A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = \frac{1! \cdot 4! \cdot 7! \cdots (3n-2)!}{n! \cdot (n+1)! \cdots (2n-1)!} \]
Richard Stanley

Andrews’ Theorem: the number of descending plane partitions of size $n$ is

$$A_n = \prod_{j=0}^{n-1} \frac{(3j + 1)!}{(n + j)!} = \frac{1! \cdot 4! \cdot 7! \cdots (3n - 2)!}{n! \cdot (n + 1)! \cdots (2n - 1)!}$$

George Andrews
All you have to do is find a 1-to-1 correspondence between $n$ by $n$ alternating sign matrices and descending plane partitions of size $n$, and conjecture 2 will be proven!
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**What is a descending plane partition?**
Percy A. MacMahon

Plane Partition

Work begun in 1897
Plane partition of 75

# of pp’s of 75 = pp(75)
Plane partition of 75

# of pp’s of 75 = \text{pp}(75) = 37,745,732,428,153
Generating function:

\[ 1 + \sum_{j=1}^{\infty} pp(j)q^j = 1 + q + 3q^2 + 6q^3 + 13q^4 + \ldots \]

\[ = \prod_{k=1}^{\infty} \frac{1}{(1 - q^k)^k} \]

\[ = \frac{1}{(1-q)(1-q^2)^2(1-q^3)^3} \ldots \]
1912 MacMahon proves that the generating function for plane partitions in an $n \times n \times n$ box is

$$\prod_{1 \leq i, j, k \leq n} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}}$$

At the same time, he conjectures that the generating function for \textit{symmetric} plane partitions is

$$\prod_{1 \leq i=j \leq n, 1 \leq k \leq n} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}} \prod_{1 \leq i < j \leq n, 1 \leq k \leq n} \frac{1 - q^{2(i+j+k-1)}}{1 - q^{2(i+j+k-2)}}$$
“The reader must be warned that, although there is little doubt that this result is correct, ... the result has not been rigorously established. ... Further investigations in regard to these matters would be sure to lead to valuable work.” (1916)
1971 Basil Gordon proves case for $n = \infty$
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1977 George Andrews and Ian Macdonald independently prove general case
1912 MacMahon proves that the generating function for plane partitions in an $n \times n \times n$ box is

\[
\prod_{1 \leq i, j, k \leq n} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}}
\]

At the same time, he conjectures that the generating function for \textbf{symmetric} plane partitions is

\[
\prod_{1 \leq i = j \leq n} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}} \prod_{1 \leq i < j \leq n} \frac{1 - q^{2(i+j+k-1)}}{1 - q^{2(i+j+k-2)}}
\]
Macdonald’s observation: both generating functions are special cases of the following

\[ B(r, s, t) = \{(i, j, k)|1 \leq i \leq r, 1 \leq j \leq s, 1 \leq k \leq t\} \]

\[ \text{ht}(i, j, k) = i + j + k - 2 \]

\[
\text{generating function} = \prod_{\eta \in B/G} \frac{1 - q^{\text{ht}(\eta)(1 + \text{ht}(\eta))}}{1 - q^{\text{ht}(\eta)}}
\]

where \( G \) is a group acting on the points in \( B \) and \( B/G \) is the set of \textit{orbits}. If \( G \) consists of only the identity, this gives all plane partitions in \( B \). If \( G \) is the identity and \((i,j,k) \rightarrow (j,i,k)\), then get generating function for symmetric plane partitions.
Does this work for other groups of symmetries?

$G = S_3$ ?
Does this work for other groups of symmetries?

$G = S_3$? No
Does this work for other groups of symmetries?

$G = S_3$? No

$G = C_3$? $(i,j,k) \rightarrow (j,k,i) \rightarrow (k,i,j)$

It seems to work.
Cyclically Symmetric Plane Partition
Macdonald’s Conjecture (1979): The generating function for cyclically symmetric plane partitions in $B(n,n,n)$ is

\[ \prod_{\eta \in B/C_3} \frac{1 - q^{1+\text{ht}(\eta)}}{1 - q^{\text{ht}(\eta)}} \]

“If I had to single out the most interesting open problem in all of enumerative combinatorics, this would be it.” Richard Stanley, review of *Symmetric Functions and Hall Polynomials*, *Bulletin of the AMS*, March, 1981.
1979, Andrews counts cyclically symmetric plane partitions
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1979, Andrews counts cyclically symmetric plane partitions
1979, Andrews counts cyclically symmetric plane partitions
1979, Andrews counts cyclically symmetric plane partitions

\[ L_1 = W_1 > L_2 = W_2 > L_3 = W_3 > \ldots \]
1979, Andrews counts **descending** plane partitions

\[ L_1 > W_1 \geq L_2 > W_2 \geq L_3 > W_3 \geq \ldots \]
6 X 6 ASM $\leftrightarrow$ DPP with largest part $\leq 6$

What are the corresponding 6 subsets of DPP’s?

\begin{align*}
6 & \quad 6 & \quad 6 & \quad 4 & \quad 3 \\
3 & \quad 3 & \quad 2
\end{align*}
ASM with 1 at top of first column $\leftrightarrow$ DPP with no parts of size $n$.

ASM with 1 at top of last column $\leftrightarrow$ DPP with $n-1$ parts of size $n$. 

```
6 6 6 4 3
3 3
2
```
Mills, Robbins, Rumsey Conjecture: # of $n \times n$ ASM’s with 1 at top of column $j$ equals # of DPP’s $\leq n$ with exactly $j-1$ parts of size $n$. 
Mills, Robbins, & Rumsey proved that # of DPP’s ≤ n with j parts of size n was given by their conjectured formula for ASM’s.
Mills, Robbins, & Rumsey proved that \# of DPP’s $\leq n$ with $j$ parts of size $n$ was given by their conjectured formula for ASM’s.

Discovered an easier proof of Andrews’ formula, using induction on $j$ and $n$. 
Mills, Robbins, & Rumsey proved that the number of DPP’s $\leq n$ with $j$ parts of size $n$ was given by their conjectured formula for ASM’s.

Discovered an easier proof of Andrews’ formula, using induction on $j$ and $n$.

Used this inductive argument to prove Macdonald’s conjecture

Mills, Robbins, & Rumsey proved that \( \# \) of DPP’s \( \leq n \) with \( j \) parts of size \( n \) was given by their conjectured formula for ASM’s.

Discovered an *easier* proof of Andrews’ formula, using induction on \( j \) and \( n \).

Used this inductive argument to prove Macdonald’s conjecture


But they still didn’t have a proof of *their* conjecture!
Totally Symmetric Self-Complementary Plane Partitions

1983

Vertical flip of ASM $\leftrightarrow$ complement of DPP?
Totally Symmetric Self-Complementary Plane Partitions
Robbins’ Conjecture: The number of TSSCPP’s in a $2n \times 2n \times 2n$ box is

$$\prod_{j=0}^{n-1} \frac{(3j + 1)!}{(n + j)!} = \frac{1! \cdot 4! \cdot 7! \cdots (3n - 2)!}{n! \cdot (n + 1)! \cdots (2n - 1)!}$$
Robbins’ Conjecture: The number of TSSCPP’s in a $2n \times 2n \times 2n$ box is

$$\prod_{j=0}^{n-1} \frac{(3j + 1)!}{(n + j)!} = \frac{1! \cdot 4! \cdot 7! \cdots (3n - 2)!}{n! \cdot (n + 1)! \cdots (2n - 1)!}$$

1989: William Doran shows equivalent to counting lattice paths

1990: John Stembridge represents the counting function as a Pfaffian (built on insights of Gordon and Okada)

1992: George Andrews evaluates the Pfaffian, proves Robbins’ Conjecture
December, 1992

Doron Zeilberger announces a proof that 
# of ASM’s of size $n$
equals of TSSCPP’s in 
box of size $2n$. 
December, 1992

Doron Zeilberger announces a proof that 
# of ASM’s of size $n$
equals of TSSCPP’s in box of size $2n$.

Zeilberger’s proof is an 84-page tour de force, but it still left open the original conjecture:

\[
\frac{A_{n,k}}{A_{n,k+1}} = \frac{\binom{n-2}{k-1} + \binom{n-1}{k-1}}{\binom{n-2}{n-k-1} + \binom{n-1}{n-k-1}}
\]
1996 Kuperberg announces a simple proof

“Another proof of the alternating sign matrix conjecture,”
International Mathematics Research Notices

Greg Kuperberg
UC Davis
1996 Kuperberg announces a simple proof

“Another proof of the alternating sign matrix conjecture,”
*International Mathematics Research Notices*

Physicists have been studying ASM’s for decades, only they call them *square ice* (aka the *six-vertex model*).
\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
\]
Horizontal → 1

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]

Vertical → -1
$N = \# \text{ of vertical}$

$I = \text{inversion number}$

$= N + \# \text{ of NE}$
1960’s
Rodney Baxter’s
Triangle-to-triangle relation
1960’s
Rodney Baxter’s
Triangle-to-triangle relation

1980’s
Anatoli Izergin

Vladimir Korepin
\[
\det \left( \frac{1}{(x_i - y_j)(ax_i - y_j)} \right) \prod_{i,j=1}^{n} \frac{(x_i - y_j)(ax_i - y_j)}{\prod_{1 \leq i < j \leq n} (x_i - x_j)(y_i - y_j)}
\]

\[
= \sum_{A \in \mathcal{A}_n} (1 - a)^{2N(A)} a^{n(n-1)/2 - \text{Inv}(A)}
\times \prod_{\text{vert}} x_i y_j \prod_{\text{SW, NE}} (ax_i - y_j) \prod_{\text{NW, SE}} (x_i - y_j)
\]
\[
\det \left( \frac{1}{(x_i - y_j)(ax_i - y_j)} \right) \prod_{i,j=1}^{n} (x_i - y_j)(ax_i - y_j) \prod_{1 \leq i < j \leq n} (x_i - x_j)(y_i - y_j)
\]

\[
= \sum_{A \in \mathcal{A}_n} (1 - a)^{2N(A)} a^{n(n-1)/2-\text{Inv}(A)}
\]

\[
\times \prod_{\text{vert}} x_i y_j \prod_{\text{SW, NE}} (ax_i - y_j) \prod_{\text{NW, SE}} (x_i - y_j)
\]

\[
a = z^{-4}, \quad x_i = z^2, \quad y_i = 1
\]

\[
\text{RHS} = \left( z - z^{-1} \right)^{n(n-1)} \sum_{A \in \mathcal{A}_n} (z + z^{-1})^{2N(A)}
\]

\[
z = e^{\pi i / 3}
\]

\[
\text{RHS} = (-3)^{n(n-1)/2} |\mathcal{A}_n|
\]
Doron Zeilberger uses this determinant to prove the original conjecture

The End

(which is really just the beginning)
The End

(which is really just the beginning)

This Power Point presentation can be downloaded from
www.macalester.edu/~bressoud/talks