HYDRO-TURBINE OPTIMIZATION

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1. Introduction

The Great Northern Paper Company in Millinocket, Maine, produces newsprint, computer paper, and many other kinds of paper goods. In order to ensure an adequate supply of affordable power, it also operates six hydro-electric generating stations on the Penobscot River. In the present problem we are concerned with the power station on the West Branch of the Penobscot River, which gets its water from a dam on Ripogenus Lake. A pipe sixteen feet in diameter and three-quarters of a mile long carries water from the dam to the power station, through an elevation drop of 170 feet. The rate at which water flows through the pipe varies, depending on conditions in the watershed.

Once at the power station, manually controlled valves and gates distribute the water to the station's three hydro-electric turbines. These turbines have known, and different, "power curves," which give the amount of electric power generated as a function of the water flow sent to the turbine. The problem, as presented to us by the power plant supervisor, is to devise a plan for distributing water among the turbines which will get the maximum energy production from the three turbines for any rate of water flow.

2. Modeling the Problem

Our plan will be to formulate the problem mathematically—build a mathematical model of the problem. We will then solve the mathematical problem and translate our results into an operational plan for distributing water to the turbines. We start with some background.

Commercial electricity is produced by turbines which turn mechanical energy into electrical current. In some cases, coal, oil, gas, or atomic fuel is used to make steam which runs the turbine. Hydro-electric power stations use the energy of falling water to turn the turbines. The energy comes both from the weight of the water and from the "head" on it, that is, the vertical distance through which the water falls.

The basic equation which relates water flow to energy production was published by Daniel Bernoulli in 1738 and is called *Bernoulli's equation*. It results from applying the principle of conservation of energy to the flow between the lake surface and the turbine. In our context, the equation states that

$$W = \gamma Q \eta (Z_h - Z_t - f), \tag{1}$$

Taken from *Applications of Calculus*, vol. 3 in *Resources for Calculus*, MAA Notes #29.

where

$$\begin{aligned} W &= \text{power extracted by turbine (foot-pounds/second)} \\ \gamma &= \text{specific weight of water (pounds/foot}^3\text{)} \\ Q &= \text{flow rate of fluid (feet}^3\text{/second, abbreviated cfs)} \\ \eta &= \text{turbine efficiency, a function of } Q \\ Z_h &= \text{elevation of the lake surface (feet)} \\ Z_t &= \text{elevation of the turbine (feet)} \\ f &= \text{energy loss due to friction, a function of } Q. \end{aligned}$$

In our case, the difference between Z_h and Z_t is 170 feet. The main factor in f is the energy lost as water flows through the pipe. Engineers derived from experiment the estimate $f = 1.6 \cdot 10^{-6} Q_T^2$, where Q_T is the total water flow in cubic feet per second (cfs).

The efficiency η, which is a function of Q, differs for the three turbines. Experimental results suggest expressing $\gamma Q \eta$ as a quadratic polynomial in Q for each turbine. Statistical curve fitting then gave the following equations for the power output of the three turbines:

$$KW_1 = \left(-18.89 + 0.1277 Q_1 - 4.08 \cdot 10^{-5} Q_1^2\right)\left(170 - 1.6 \cdot 10^{-6} Q_T^2\right), \quad 250 \le Q_1 \le 1110, \tag{2}$$

$$KW_2 = \left(-24.51 + 0.1358 Q_2 - 4.69 \cdot 10^{-5} Q_2^2\right)\left(170 - 1.6 \cdot 10^{-6} Q_T^2\right), \quad 250 \le Q_2 \le 1110, \tag{3}$$

$$KW_3 = \left(-27.02 + 0.1380 Q_3 - 3.84 \cdot 10^{-5} Q_3^2\right)\left(170 - 1.6 \cdot 10^{-6} Q_T^2\right), \quad 250 \le Q_3 \le 1225, \tag{4}$$

where

$$\begin{aligned} Q_i &= \text{flow through turbine } i \text{ (cfs)} \\ KW_i &= \text{power generated by turbine } i \text{ (kilowatts)} \\ Q_T &= \text{total flow through the station (cfs).} \end{aligned}$$

The coefficients in the quadratic polynomials in (2), (3), and (4) include a scaling factor to transform units of mechanical power into units of kilowatts. The bounds on the Q_i represent the fact that the turbines cannot operate with a flow below 250 cfs, or above a maximum flow which is slightly higher for turbine 3 than for turbines 1 and 2.

If all three turbines are running, our problem in distributing water among the turbines to obtain the maximum energy production can now be formulated as a mathematical problem:

$$\text{Maximize } KW_1 + KW_2 + KW_3 \tag{5}$$

subject to

$$Q_1 + Q_2 + Q_3 = Q_T \quad \text{and}$$
$$250 \le Q_1 \le 1110, \quad 250 \le Q_2 \le 1110, \quad 250 \le Q_3 \le 1225.$$

We must solve this problem for all feasible values of Q_T.

3. Solving the Constrained Optimization Problem

Problem 1. *Set up the system of equations with a Lagrange multiplier that can be used to solve this optimization problem.*

Problem 2. *What is the solution of this constrained problem, giving Q_1, Q_2 and Q_3 in terms of Q_T? If $Q_T = 2500$ cfs, what are the values of Q_1, Q_2, and Q_3?*

Problem 3. *What are the limits on Q_T for which this solution is valid?*

The solution to Problem 2 is not a complete solution to the original problem because it does not deal with how to distribute the total flow Q_T when it is outside the limits found in Problem 3. Also, we have assumed that we should run all three turbines. Perhaps there is a range of values when we could run all three turbines, but when it might be more efficient to only run one or two turbines.

If we were going to run just one or two turbines, we would want them to be the most efficient ones. The *efficiency* of a turbine is the number of kilowatts produced per unit of flow: KW/Q. Since we only need to compare the turbines to each other, we can ignore the constant factor $C = 170 - 1.6 \cdot 10^{-6} Q_T^2$.

Problem 4. *Graph the efficiencies of each of the three turbines: $KW/(QC)$ as a function of Q. As the flow increases, each turbine rises to a peak efficiency and then declines slightly in efficiency if the flow continues to increase. Find the intervals between the values of Q where the graphs cross and on each of these intervals, list the turbines in order of most to least efficient.*

If we run just one turbine, it should be the most efficient for that range of values of Q, It is then easy to calculate how much energy is produced by that turbine for each value of Q in that range.

Problem 5. *Find the amount of energy produced at each value of Q_T, $250 \leq Q_T \leq 1225$, if we run only the most efficient turbine for that value of Q.*

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If we run two turbines, then we should shut down the least efficient turbine for that range of values. To find out how much energy we can get from these two turbines, we need to solve a new constrained maximization problem, again using Lagrange multipliers.

Problem 6. *For each range of values of Q over which the least efficient turbine does not change, solve the constrained optimization problem to find how much energy can be produced at each value of Q using the two most efficient turbines.*

Problem 7. *Compare the power outputs using one, two, or three turbines in each range of values. How should the total flow be distributed for each value of Q_T?*

At this point, we need to give some thought to how we should present our solution. It must be in a form that can be used by the plant operator who controls flow to the turbines by valves and gates. In this context, a table giving the suggested distribution of flow for different values of Q_T would be more useful than mathematical formulas. Consultation with the plant operator suggests that, given the accuracy of the flow gauges, a table giving the distribution as Q_T increases in increments of 100 cfs would be satisfactory.

The method that the plant has been using has the advantage of simplicity:

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\begin{array}{rl}
250 \leq Q_T \leq 500, & \text{use turbine 1 only} \\
500 < Q_T \leq 750, & \text{divide the flow evenly between turbines 1 and 2} \\
750 < Q_T \leq 3330, & \text{divide the flow evenly among all three turbines,} \\
3330 < Q_T \leq 3445, & \text{run turbines 1 and 2 at full flow and send additional flow to turbine 3.}
\end{array}
$$

Write up a full report of how you arrived at your answer and compare your solution to current practice. Include a separate self-contained summary that can be given to the plant operator.