ROBINSON-SCHENsted-KNUTH INSERTION AND CHARACTERS OF CYCLOTOMIC HECKE ALGEBRAS

ANDY CANTRELL, TOM HALVERSON, AND BRIAN MILLER


0. Introduction

The cyclotomic Hecke algebras $H_{n,r} = H_n(u_1, \ldots, u_r; q)$ were defined by Ariki and Koike in [AK] as Iwahori-Hecke algebras of the complex reflection group $G_{n,r} = S_n \wr (\mathbb{Z}/r\mathbb{Z})^n$ where $S_n$ is the symmetric group. If $\zeta$ is a primitive complex $r$th root of unity, then when $q \to 1$ and $u_i \to \zeta^i$, the algebra $H_{n,r}$ specializes to the group algebra $\mathbb{C}[G_{n,r}]$. The irreducible representations of $H_{n,r}$ are constructed in [AK]. They are indexed by the set of all $r$-tuples of partitions with a total of $n$ boxes, called $r$-partitions.

For each $r$-partition $\mu$, T. Shoji [Sho] defines a symmetric function $q_\mu$ and proves that

$$q_\mu = \sum_{\lambda} \chi^\lambda_q(a_\mu) s_\lambda,$$

where $s_\lambda$ is the Schur function associated to the $r$-partition $\lambda$ and $\chi^\lambda_q(a_\mu)$ is the irreducible $H_{n,r}$-character associated to $\lambda$ and evaluated at an element $a_\mu$. The function $q_\mu$ is a deformation of the power sum symmetric function, and Shoji’s formula is analogous to the Frobenius formula for symmetric group characters. Shoji proves it using the Schur-Weyl duality for $H_{n,r}$ found in [SS].

In this paper we derive the formula

$$q_\mu = \sum_{\lambda \in P_{n,r}} \left( \sum_{Q_\lambda} wt_\mu(Q_\lambda) \right) s_\lambda,$$

where $Q_\lambda$ ranges over the set of “standard tableaux” of shape $\lambda$, and where $wt_\mu$ is a weight on standard tableaux that depends on the parameters $q$ and $u_i$ and that is computed combinatorially. By comparing coefficients of $s_\lambda$ in these two formulas we obtain the expression

$$\chi^\lambda_q(a_\mu) = \sum_{Q_\lambda} wt_\mu(Q_\lambda)$$

which computes the irreducible $H_{n,r}$-characters as a sum over standard tableaux. When $q = 1$ and $u_i = \zeta^i$, our character formula specializes to a character formula for the complex reflection group $G_{n,r}$.

In the special case where $n = r = 1$, the cyclotomic Hecke algebra $H_{1,1}$ is the Iwahori-Hecke algebra $H_n(q)$ of type $A_{n-1}$ associated with the symmetric group $S_n$. Shoji’s Frobenius formula specializes, in this case, to the Frobenius formula

---

Research supported in part by National Science Foundation grant DMS-0100975.
of A. Ram [Ra1] for \( H_n(q) \) and our character formula is a generalization of the Roichman formula [Ro] for irreducible characters of \( H_n(q) \) and \( S_n \).

Our method is to follow the work of Ram [Ra2] who gives a new proof of the Roichman formula for \( H_n(q) \) using Robinson-Schensted-Knuth insertion. We write the function \( q_\mu \) as a sum over \( \mu \)-weighted integer sequences. We then use RSK insertion, modified for \( r \)-partitions, to turn this into a sum over pairs \((P, Q)\) where \( P \) is a column-strict tableau, \( Q \) is a standard tableau, and \( P \) and \( Q \) have the same shape \( \lambda \) for some \( r \)-partition \( \lambda \). As a special case of our insertion rule we obtain a bijective proof of the formula

\[
n!r^n = \sum_\lambda f_\lambda^2
\]

where \( n!r^n = |G_{n,r}| \) and \( f_\lambda \) is the number of standard tableau whose shape is the \( r \)-partition \( \lambda \). This fact can be proved algebraically by decomposing the regular representation of \( G_{n,r} \) into irreducibles and comparing dimensions.

A Murnaghan-Nakayama type rule for the characters of \( H_n,q \) is found in [HR]. It gives the irreducible characters of \( H_n,q \) as weighted sums over broken-border-strip tableaux. The characters \( \chi_\lambda^\mu(q_\mu) \) found in Shoji’s Frobenius formula and in this paper are evaluated on a set \( \{a_\mu\} \) of elements in \( H_{n,r} \) for which characters are completely determined. The character values found in [HR] are evaluated on different elements \( T_\mu \).

1. Cyclotomic Hecke Algebras

Let \( u_1, \ldots, u_r \) and \( q \) be indeterminates. The cyclotomic Hecke algebra \( H_{n,r} = H_n(u_1, \ldots, u_r; q) \) is the algebra over \( \mathbb{C}(q, u_1, \ldots, u_r) \) defined by generators \( X_1, T_1, \ldots, T_{n-1} \), and relations

\[
\begin{align*}
(1) \quad & T_i^q = (q - q^{-1})T_i + 1, & 1 \leq i \leq n - 1, \\
(2) \quad & T_iT_j = T_jT_i, & |i - j| > 1, \\
(3) \quad & T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, & 1 \leq i \leq n - 2, \\
(4) \quad & X_1T_1X_1T_1 = T_1X_1T_1X_1, \\
(5) \quad & (X_1 - u_1)(X_1 - u_2) \cdots (X_1 - u_r) = 0.
\end{align*}
\]

These algebras were introduced by Ariki and Koike [AK], and they are semisimple over \( \mathbb{C}(q, u_1, \ldots, u_r) \).

Let \( S_n \) be the symmetric group on \( n \) letters, and let \( G_{n,r} = S_n \wr (\mathbb{Z}/r\mathbb{Z})^n \). The group \( G_{n,r} \) has a presentation on generators \( t_1, s_1, \ldots, s_{n-1} \) where \( t_1^r = 1 \) and \( s_1, \ldots, s_{n-1} \) are the simple transpositions in \( S_n \). If we let

\[
q \to 1, \quad u_i \to \zeta^i (1 \leq i \leq r), \quad T_i \to s_i (1 \leq i \leq n - 1), \quad \text{and} \quad X_1 \to t_1,
\]

where \( \zeta = \) a primitive \( r \)th root of unity in \( \mathbb{C} \),

then the presentation for \( H_{n,r} \) above becomes a presentation for \( \mathbb{C}[G_{n,r}] \).

1.1. \( r \)-partitions.

We use the usual notation for partitions found in [Mac]. We identify a a partition with its Young diagram, let \( \ell(\lambda) \) denote the number of rows of \( \lambda \), and \( |\lambda| \) denote the number of boxes in \( \lambda \). For example, \( \lambda = (5, 5, 3, 1) \) has \( \ell(\lambda) = 5 \) and \( |\lambda| = 15 \).

An \( r \)-tuple of partitions \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \) is called an \( r \)-partition. We refer to the \( \lambda^{(k)} \) as the components of \( \lambda \). We let \( |\lambda| = \sum_{k=1}^r |\lambda^{(k)}| \) denote the total number
of boxes in $\lambda$, and we let $\ell(\lambda) = \sum_{k=1}^\ell(\lambda^{(k)})$ denote the total number of rows in $\lambda$. If $|\lambda| = n$, then we say that $\lambda$ is an $r$-partition of $n$. For example, if $r = 5$, then

$$\lambda = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{has } \ell(\lambda) = 11 \text{ and } |\lambda| = 24,$$

and, for example, $\lambda^{(2)} = (3, 3, 1, 1)$. We let $\mathcal{P}_{n,r}$ denote the set of all $r$-partitions of $n$.

### 1.2. Irreducible Representations and Characters.

It is known by [AK] that the irreducible representations of $H_{n,r}$ are indexed by $\mathcal{P}_{n,r}$. We let $V^\lambda_q$ denote the irreducible $H_{n,r}$-module corresponding to $\lambda \in \mathcal{P}_{n,r}$, and we let $\chi^\lambda_q$ denote the corresponding irreducible character. The irreducible representations and characters of $G_{n,r}$ are also indexed by $\mathcal{P}_{n,r}$. We denote them by $V^\lambda_1$ and $\chi^\lambda_1$. The construction of $V^\lambda_q$ in [AK] is such that when $q = 1$ and $u_i = \xi^i$, $V^\lambda_q$ becomes $V^\lambda_1$ and $\chi^\lambda_q$ becomes $\chi^\lambda_1$.

### 1.3. Standard Elements.

The conjugacy classes of $G_{n,r}$ are also parameterized by $\mathcal{P}_{n,r}$. Define $t_k = s_k s_{k-1} \cdots s_1 t_1 s_{k-1} \cdots s_1$ for $2 \leq k \leq n$, and define

$$w(1, i) = t_1^i \quad \text{and} \quad w(k, i) = t_k^i s_{k-1} \cdots s_1, \quad 2 \leq k \leq n.$$ 

For a partition $\mu = (\mu_1, \ldots, \mu_\ell)$ with $|\mu| = n$, define

$$w(\mu, i) = w(\mu_1, i) \times \cdots \times w(\mu_\ell, i)$$

with respect to the embedding $G_{\mu_1, r} \times \cdots \times G_{\mu_\ell, r} \subseteq G_{n, r}$. For $\mu \in \mathcal{P}_{n,r}$, define

$$w_{\mu} = w(\mu^{(1)}, 1) w(\mu^{(2)}, 2) \cdots w(\mu^{(r)}, r).$$

Then $\{w_{\mu} | \mu \in \mathcal{P}_{n,r}\}$ is a set of conjugacy class representatives for $G_{n,r}$. T. Shoji [Sho] defines elements $\xi_1, \ldots, \xi_n \in H_{n, r}$ and shows that the elements $T_1, \ldots, T_{n-1}, \xi_1, \ldots, \xi_n$ generate $H_{n, r}$ in [Sho], §3.6, he gives a presentation of $H_{n, r}$ in terms of generators $T_1, \ldots, T_{n-1}, \xi_1, \ldots, \xi_n$ and relations. The relation between $X_1$ and the $\xi_i$ is ????? Define

$$a(1, i) = \xi_1^i \quad \text{and} \quad a(k, i) = \xi_k^i T_{k-1} \cdots T_1, \quad 2 \leq k \leq n.$$ 

For a partition $\mu = (\mu_1, \ldots, \mu_\ell)$ with $|\mu| = n$, define

$$a(\mu, i) = a(\mu_1, i) \times \cdots \times a(\mu_\ell, i)$$

with respect to the embedding $H_{\mu_1, r} \otimes \cdots \otimes H_{\mu_\ell, r} \subseteq H_{n, r}$. For $\mu \in \mathcal{P}_{n,r}$, define

$$a_{\mu} = a(\mu^{(1)}, 1) a(\mu^{(2)}, 2) \cdots a(\mu^{(r)}, r).$$

Shoji [Sho], Proposition 7.5, proves that any character of $H_{n, r}$ is completely determined by its value on the set $\{a_{\mu} | \mu \in \mathcal{P}_{n,r}\}$. 
2. Symmetric Functions

In this section, we follow [Mac], Appendix B, and [Sho] and define symmetric
functions indexed by $r$-partitions.

Let $m_1, \ldots, m_r$ be positive integers satisfying $m_k \geq n$ for each $1 \leq k \leq r$, and
let $m = \sum_{k=1}^{r} m_k$. We define a set $x$ of $m$ indeterminates as follows
\[ x^{(k)} = \{x_{1}^{(k)}, \ldots, x_{m_k}^{(k)}\}, \quad 1 \leq k \leq r, \]
\[ x = x^{(1)} \cup \cdots \cup x^{(r)}. \]
We say that the indeterminates in $x^{(k)}$ are of color $k$, and we linearly order the
indeterminates $x = x_1^{(1)}, \ldots, x_m^{(r)}$ by the rule,
\[ x_i^{(k)} < x_j^{(\ell)} \text{ if and only if } k < \ell \text{ or } k = \ell \text{ and } i < j. \]

It is sometimes notationally convenient to identify the variables $x = x_1^{(1)}, \ldots, x_m^{(r)}$
with the variables $x = x_1, \ldots, x_m$ as follows,

\[
\begin{array}{cccccccc}
  x_1, & x_2, & \ldots, & x_m, & x_{m_1+1}, & x_{m_1+2}, & \ldots, & x_m,
  \\
  x_1^{(1)}, & x_2^{(1)}, & \ldots, & x_{m_1}^{(1)}, & x_1^{(2)}, & x_2^{(2)}, & \ldots, & x_m^{(r)}.
\end{array}
\]

To do this explicitly, set $x_j = x_j^{(b(j))}$, with $d_j = \sum_{i=1}^{b(j)} m_i$, and we define a function
\[ b(j) = k, \text{ where } m_1 + \ldots + m_k < j \leq m_1 + \ldots + m_{k+1}, \]
so that $b(j)$ gives the color of the indeterminate $x_j$. We will interchangeably use these
two notations for variables.

Recall from Section 1, that $\zeta$ is a primitive $r$th root of unity in $\mathbb{C}$. For integers
$t \geq 1$ and $1 \leq i \leq r$, let
\[ p_i^{(j)}(x) = \sum_{j=1}^{r} \zeta^{ij} p_i(x^{(j)}), \]
where $p_i(x^{(j)})$ denotes the $i$th power sum symmetric function ([Mac], I§2) with
respect to the variables $x^{(j)}$. As a special case, we let $p_0^{(j)}(x) = 1$ for each $i$. For
$\mu \in \mathcal{P}_{n,r}$ with $\mu = (\mu^{(1)}, \ldots, \mu^{(r)})$ and $\mu^{(k)} = (\mu_1^{(k)}, \ldots, \mu_{\ell_k}^{(k)})$, define
\[ p_\mu(x) = \prod_{k=1}^{r} \prod_{j=1}^{\ell_k} p_{\mu_{j}^{(k)}}^{(k)}(x). \]

Definition (2.5) is given in [Sho] and it is the complex conjugate of the definition of
$p_\mu$ given in [Mac].

Now we define the Schur function associated to $\lambda \in \mathcal{P}_{n,r}$ by
\[ s_\lambda(x) = \prod_{k=1}^{r} s_{\lambda^{(k)}}(x^{(k)}), \]
where $s_{\lambda^{(k)}}(x^{(k)})$ denotes the Schur function ([Mac], I§3) associated to the partition
$\lambda^{(k)}$ with respect to the variables $x^{(k)}$. If $\lambda \in \mathcal{P}_{n,r}$, then a column-strict tableau of
shape $\lambda$ is a filling of the boxes of $\lambda$ with integers such that for each $k$
(1) $\lambda^{(k)}$ contains integers from the set $\{1, \ldots, m_k\}$,
(2) the columns of $\lambda^{(k)}$ strictly increase from top to bottom, and
(3) the rows of $\lambda^{(k)}$ weakly increase (do not decrease) from left to right.
For example,
\[
\begin{pmatrix}
1 & 1 & 1 \\
2 & 3 \\
3
\end{pmatrix},
\begin{pmatrix}
1 & 1 & 2 \\
2 & 4 \\
3 & 5 \\
6
\end{pmatrix},
\emptyset,
\begin{pmatrix}
1 & 4 \\
2 & 5 \\
3 & 6 \\
7
\end{pmatrix},
\begin{pmatrix}
1 & 3 & 3 \\
4 & 5 & 6
\end{pmatrix}
\]
is a column-strict tableau of shape $\lambda$.

For a column-strict tableau $P_\lambda$ of shape $\lambda$ we define
\[
x^{P_\lambda} = \prod_{k=1}^{r} \prod_{j=1}^{m_k} (x^{(k)}_j)^{m_{jk}(P_\lambda)},
\]
where $m_{jk}(P_\lambda)$ denotes the number of times that $j$ appears in the $k$th component (i.e., $\lambda^{(k)}$) of $P_\lambda$. It follows from [Mac] I.5.12 that
\[
s_\lambda(x) = \sum_{P_\lambda} x^{P_\lambda},
\]
where the sum is over all column-strict tableaux $P_\lambda$ of shape $\lambda$.

We now define a deformation of $p_\mu$. Let $u$ denote the parameters $u_1, \ldots, u_r$. For integers $t \geq 1$ and $1 \leq i \leq r$, let
\[
q_t^{(i)}(x; q, u) = \sum_{I=(i_1, \ldots, i_t)} u_b^{(\max(I))} e(I) (q - q^{-1})^{\ell(I)} x_{i_1} x_{i_2} \cdots x_{i_t},
\]
where $e(I)$ is the number of $i_j \in I$ such that $i_j = i_{j+1}$, $\ell(I)$ is the number of $i_j \in I$ such that $i_j < i_{j+1}$, $\max(I)$ is the maximum element of $I$, and $b$ is the function defined in (2.3). This definition of $q_t^{(i)}$ is given in [Sho]. For $\mu \in P_{n,r}$ with $\mu = (\mu^{(1)}, \ldots, \mu^{(r)})$ and $\mu^{(k)} = (\mu^{(k)}_1, \ldots, \mu^{(k)}_{t_k})$, define
\[
q_\mu(x; q, u) = \prod_{k=1}^{r} \prod_{j=1}^{t_k} q_{u_k}^{(k)}(x; q, u).
\]
Note that when $q = 1$ and $u_i = \zeta^i$, we have $q_\mu = p_\mu$.

In [Mac], Appendix B, (9.7), we find the following Frobenius formula for the irreducible characters of $G_{n,r}$,
\[
p_\mu(x) = \sum_{\lambda \in P_{n,r}} \chi^{\lambda}(w_\mu) s_\lambda(x),
\]
for each $\mu \in P_{n,r}$. Shoji [Sho] extends this formula to a Frobenius formula for the irreducible characters of $H_{n,r}$,
\[
q_\mu(x; q, u) = \sum_{\lambda \in P_{n,r}} \chi^{\lambda}(a_\mu) s_\lambda(x),
\]
for each $\mu \in P_{n,r}$.

We say that $I = (i_1, \ldots, i_t)$ is an up-down sequence if there exists an $s$, with $0 \leq s \leq t$, such that
\[
i_1 < \cdots < i_s < i_{s+1} \geq \cdots \geq i_t,
\]
for some $s$, with $0 \leq s < t$,
and we say that $i_{s+1}$ is the peak of the up-down sequence $I$. Note that any of $i_1, \ldots, i_t$ can potentially be the peak of an up-down sequence $I = (i_1, \ldots, i_t)$.
Following [Ra2], we define the weight

\[(2.13) \quad \text{wt}(i_1, \ldots, i_t) = \begin{cases} 0, & \text{if } i_1, \ldots, i_t \text{ is not an up-down sequence} \\ (-q)^{s-1}q^{t-1-s}, & \text{if } i_1 < \cdots < i_s < i_{s+1} \geq \cdots \geq i_t. \end{cases} \]

If \( t = 1 \) the weight is \( \text{wt}(i_1) = 1 \).

**Lemma 2.1.** [Ra2] Let \( I = (i_1, \ldots, i_t) \) with \( 1 \leq i_1 \leq i_2 \leq \cdots \leq i_t \leq m \), and let \( S_I \) denote the set of all distinct permutations of \( I \). Then

\[ q^{e(I)}(q - q^{-1})^{t(I)} = \sum_{\sigma \in S_I} \text{wt}(\sigma I) \]

where \( e(I) \) is the number of \( i_j \in I \) such that \( i_j = i_{j+1} \) and \( t(I) \) is the number of \( i_j \in I \) such that \( i_j < i_{j+1} \).

**Proof.** In [Ra2], Lemma 1.5, Ram proves the first equality below

\[
\sum_{I=(i_1, \ldots, i_t)} q^{e(I)}(q - q^{-1})^{t(I)} x_{i_1} \cdots x_{i_t} = \sum_{I=(i_1, \ldots, i_t)} \text{wt}(I) x_{i_1} \cdots x_{i_t} = \sum_{s=1}^{i_t} \sum_{\sigma \in S_I} \text{wt}(\sigma I) x_{i_1} \cdots x_{i_t}.
\]

The second equality follows from the fact that \( x_{i_1} \cdots x_{i_t} = x_{i_{\sigma(1)}} \cdots x_{i_{\sigma(t)}} \) for all \( \sigma \in S_I \). The result is obtained by comparing coefficients of \( x_{i_1} \cdots x_{i_t} \). \( \square \)

**Proposition 2.2.** For integers \( t \geq 1 \) and \( 1 \leq k \leq r \), we have

\[ q^{t(k)}(x; q, u) = \sum_{i_1 < \cdots < i_s < i_{s+1} \geq \cdots \geq i_t} (-q)^{s}q^{t-1-s}u^{k}_{i} x_{i_1} \cdots x_{i_t}, \]

where the sum is over all up-down sequences with \( i_1 < \cdots < i_s < i_{s+1} \geq \cdots \geq i_t \) and \( 1 \leq i_j \leq m \).

**Proof.** As in Lemma 2.1, let \( S_I \) denote the set of distinct permutations of \( I \). For all \( \sigma \in S_I \) we have \( \max(I) = \max(\sigma(I)) \) and \( x_{i_1} \cdots x_{i_t} = x_{i_{\sigma(1)}} \cdots x_{i_{\sigma(t)}} \). Furthermore, if \( I \) is an up-down sequence then its peak is \( \max(I) = i_{s+1} \).

Since \( \text{wt}_q \) is 0 on sequences that are not up-down, we can first write

\[
\sum_{I=(i_1, \ldots, i_t)} \text{wt}(I)u^{k}_{i} x_{i_1} \cdots x_{i_t} = \sum_{I=(i_1, \ldots, i_t)} \text{wt}(I)u^{k}_{i} x_{i_1} \cdots x_{i_t},
\]

where the left sum is over up-down sequences and the right sum is over arbitrary sequences \( I = (i_1, \ldots, i_t) \) with \( 1 \leq i_t \leq m \). Now, we use Lemma 2.1 to write the
sum over non-decreasing sequences
\[
\sum_{I=(i_1, \ldots, i_t)} \text{wt}(I) u_{b_{\ell(I)}(I)}^k x_{i_1} \cdots x_{i_t} = \sum_{I=(i_1, \ldots, i_t)} \sum_{\sigma \in S_I} \text{wt}(\sigma I) u_{b_{\ell(I)}(\sigma I)}^k x_{i_{\sigma(1)}} \cdots x_{i_{\sigma(t)}}
\]
\[
= \sum_{I=(i_1, \ldots, i_t)} u_{b_{\ell(I)}(I)}^k x_{i_1} \cdots x_{i_t} \sum_{\sigma \in S_I} \text{wt}(\sigma I)
\]
\[
= \sum_{I=(i_1, \ldots, i_t)} q^{\ell(I)}(q - q^{-1})^{\ell(I)} u_{b_{\ell(I)}(I)}^k x_{i_1} \cdots x_{i_t},
\]
by the definition (2.9) of \( q^k \).

Let \( \mu \in \mathcal{P}_{n,r} \). The row reading tableau \( R_\mu \) of shape \( \mu \) is the \( r \)-partition \( \mu \) with the boxes filled in with the numbers \( 1, \ldots, n \) so that \( \mu^{(1)} \) contains the numbers \( 1, \ldots, \lfloor \mu^{(1)} \rfloor \) in order from left-to-right and top-to-bottom, \( \mu^{(2)} \) contains the numbers \( \lfloor \mu^{(1)} \rfloor + 1, \ldots, \lfloor \mu^{(1)} \rfloor + \lfloor \mu^{(2)} \rfloor \) in order from left-to-right and top-to-bottom, and so on. For \( 1 \leq i \leq n \) we define the component function \( c_{R_\mu}(i) \) by

\[
\begin{align*}
\text{if } i = k, & \quad \text{if } i \text{ is in the } k\text{th component of } R_\mu. \\
\text{if } k, k+1, \ldots, k+t \text{ is a row of } R_\mu, & \quad \text{then} \\
\text{the subsequence } i_k, i_{k+1}, \ldots, i_{k+t} & \quad \text{is an up-down sequence,} \\
\text{i.e., } i_k < i_{k+1} < \cdots < i_t & \quad \text{if } i \geq i_{k+t}. \\
\end{align*}
\]

The index \( i_\mu \), shown above, is the peak of the row. When \( I \) is a \( \mu \)-up-down sequence, we let \( P_\mu \) denote the set of peaks \( i_\mu \) in \( I \), one for each row of \( R_\mu \). We define the \( \mu \)-weight of a sequence \( I = (i_1, \ldots, i_n) \) by

\[
\text{wt}_\mu(I) = \begin{cases} 
0, & \text{if } I \text{ is not a } \mu\text{-up-down sequence,} \\
(-q^{-1})^{\ell(I)} q^{\gamma(I)} \prod_{i_\mu \in P_\mu} c_{R_\mu}(i_\mu), & \text{if } I \text{ is a } \mu\text{-up-down sequence,}
\end{cases}
\]

where \( \gamma(I) \) is the number of \( i_j \geq i_{j+1} \) with \( j \) and \( j+1 \) in the same row of \( R_\mu \) and \( \ell(I) \) is the number of \( i_{j} < i_{j+1} \) with \( j \) and \( j+1 \) in the same row of \( R_\mu \).

**Example 2.3.** Let \( \mu = ((5, 1), (3, 3, 1, 1), 0, (2, 2, 2), (4)) \). The row reading tableau of shape \( \mu \) is

\[
R_\mu = \begin{pmatrix}
6 & 12 & 15 & 18 & 21 & 22 & 23 & 24 \\
7 & 8 & 9 & 10 & 11 & 12 & 13 \\
14 & \emptyset & 15 & 16 & 17 & 18 & 19 & 20 \end{pmatrix}
\]

The following sequence is a \( \mu \)-up-down sequence

\[
I = [7, 11, 12, 4][10][48, 70, 75][75, 75, 30][1][50][72, 25][16, 18][119, 97][5, 80, 79, 25].
\]
The braces group the components elements according to the rows of $R$, the vertical bars indicate the separation between the components of $R$, and the numbers in boldface are the peaks of their rows. The $\mu$-weight of $I$ is

$$\text{wt}_\mu(I) = ((-q^{-1})^2q^2u_1)u_5((-q^{-1})^2u_4^2q^2u_4^2)u_5^2(qu_4^4)(-q^{-1}u_1^4)(qu_4^4)(-q^{-1}q^2u_4^4)$$

$$= (-q^{-1})^2q^2u_1^2u_3u_4^2u_5^3.$$

\[\square\]

The definition (2.10) of $q_\mu$ can be thought of as a product of $q_i^{(k)}$ over the rows of $R$, where $t$ is the length of the row and $k$ is the component of the row. Thus the following colollary is immediate from Proposition 2.2.

**Corollary 2.4.** For $\mu \in P_{n,r}$,

$$q_\mu(x; q, u) = \sum_{i_1, \ldots, i_n} \text{wt}_\mu(i_1, \ldots, i_n) x_{i_1} \cdots x_{i_n},$$

where the sum is over all $\mu$-up-down sequences $(i_1, \ldots, i_n)$ and $\text{wt}_\mu$ is defined in (2.15).

3. **RSK Insertion and Roichman Weights**

If $\lambda \in P_{n,r}$, then a standard tableau $Q_\lambda$ of shape $\lambda$ is a filling of the boxes of $\lambda$ with integers from $\{1, 2, \ldots, n\}$ such that each integer from $\{1, 2, \ldots, n\}$ appears in $Q_\lambda$ exactly once, and for each $1 \leq k \leq r$

1. the columns of $\lambda^{(k)}$ strictly increase from top to bottom, and
2. the rows of $\lambda^{(k)}$ strictly increase from left to right.

The Robinson-Schensted-Knuth (RSK) insertion scheme [Sta] is an algorithm which gives a bijection between sequences $x_{i_1}, \ldots, x_{i_n}$, with $1 \leq i_j \leq m$, and pairs $(P, Q)$ where $P$ is a column-strict tableau, $Q$ is a standard tableau, and $P$ and $Q$ have shape $\lambda$ for some partition $\lambda$ with $n$ boxes. The RSK insertion algorithm constructs the pair of tableaux $(P, Q)$ iteratively,

$$(\emptyset, \emptyset) = (P_0, Q_0), (P_1, Q_1), \ldots, (P_n, Q_n) = (P, Q),$$

in such a way that

1. $P_j$ is a column strict tableau that contains $j$ boxes, and $Q_j$ is a standard tableau that has the same shape as $P_j$,
2. $P_j$ is obtained from $P_{j-1}$ by column inserting $i_j$ into $P_{j-1}$, denoted $P_j = P_{j-1} \leftarrow i_j$,
3. $Q_j$ is obtained from $Q_{j-1}$ by putting $j$ in the newly added box (i.e., the box created in going from $P_{j-1}$ to $P_j$).

The standard tableau $Q$ is called the recording tableau.

We extend the RSK algorithm to work for tableaux whose shape are $r$-partitions. Given a sequence $x_{(k_1)}, x_{(k_2)}, \ldots, x_{(k_n)}$, with $1 \leq k_j \leq r$ and $1 \leq i_j \leq m_{k_j}$, we construct a sequence $(\emptyset, \emptyset) = (P_0, Q_0), \ldots, (P_n, Q_n) = (P, Q)$, where $P_t$ is a column-strict tableau, $Q_t$ is a standard tableau, and $P_t$ and $Q_t$ have the same $r$-partition shape. We insert $x_{(k_i)}$ into a semistandard tableau $P_{j-1}$ having $r$-partition shape as follows

$$(3.1) \quad (P_{j-1}^{(1)}, \ldots, P_{j-1}^{(r)}) \leftarrow x_{(k_i)} = (P_{j-1}^{(1)}, \ldots, P_{j-1}^{(k_i)} \leftarrow x_{(k_i)}^{(k_i)}, \ldots, P_{j-1}^{(r)}).$$
where we use usual column insertion to insert variables of type $k$ into the $k$th component of $P_{j-1}$.

For example, if $r = 3$ the result of inserting $x_2^{(1)}, x_1^{(2)}, x_4^{(2)}, x_1^{(3)}$ is

$$P_i : (\emptyset, \emptyset, \emptyset), (\{1\}, \emptyset, \emptyset), (\{2\}, 1, 0), (\{2\}, 1, 1), (\{2\}, 1, 1, 0), (\{2\}, 1, 1, 0),$$

$$Q_i : (\emptyset, \emptyset, \emptyset), (\{1\} 2, \emptyset), (\{1\} 2, \emptyset), (\{1\} 2, 1, 0), (\{1\} 2, 1, 1, 0),$$

To see that this insertion provides a bijection, we can construct the inverse algorithm by using usual column uninsertion, in the reverse order of the entries of $Q$, and using the component of $P$ to tell us the type of the uninserted variable. We denote this bijection by

$$(P, Q) \xrightarrow{\text{RSK}} x_{i_1}^{(k_1)}, \ldots, x_{i_n}^{(k_n)}$$

Let $Q_\lambda$ be a standard tableau of shape $\lambda \in \mathcal{P}_{n,r}$. If $a$ and $b$ are entries of $Q_\lambda$, we say that

$a$ is southwest of $b$ (denoted $a \preceq Q b$) if

$$a \in \lambda(k), b \in \lambda(\ell), \text{ and } k > \ell,$$

or

$a$ is south (below) and/or west (left) of $b$ in $\lambda(k)$,

$a$ is northeast of $b$ (denoted $a \succeq Q b$) if

$$a \in \lambda(k), b \in \lambda(\ell), \text{ and } k < \ell,$$

or

$a$ is north (above) and/or east (right) of $b$ in $\lambda(k)$.

In the ordering on our indeterminates, we have $x_i^{(k)} < x_j^{(\ell)}$ if $k < \ell$ or $k = \ell$ and $i < j$. The following proposition is an immediate consequence of this fact and well-known facts about RSK insertion (see [Sta] or [Ra2], Proposition 2.1).

**Proposition 3.1.** Let $P_{j+1} = (P_{j-1} \leftarrow x_{i_{j+1}}^{(k_j)}) \leftarrow x_{i_{j+1}}^{(k_j)}$, where $P_{j-1}$ is a column-strict tableau, and let $Q_{j+1}$ be the associated recording tableau.

1. If $x_{i_{j+1}}^{(k_j)} < x_{i_{j+1}}^{(k_{j+1})}$ then $j + 1$ appears southwest ($\preceq Q$) of $j$ in $Q_{j+1}$.
2. If $x_{i_{j+1}}^{(k_j)} \geq x_{i_{j+1}}^{(k_{j+1})}$ then $j + 1$ appears northeast ($\succeq Q$) of $j$ in $Q_{j+1}$.

Let $\mu, \lambda \in \mathcal{P}_{n,r}$. We say that a standard tableau $Q_\lambda$ of shape $\lambda$ is a $\mu$-SW-NE tableau if it satisfies the following property

$$k < (k + 1) \leq \cdots \leq p \geq \cdots \geq (k + t) \text{ in } Q_\lambda$$

The index $p$, shown above, is the turn-around point of the row. When $I$ is a $\mu$-up-down sequence, we let $P_{Q\lambda}$ denote the set of turn-arounds $p$ in $Q_\lambda$, one for each row of $R_\mu$.

We define the $\mu$-weight of a standard tableau $Q_\lambda$ by

$$\text{wt}_\mu(Q_\lambda) = \begin{cases} 0, & \text{if } Q_\lambda \text{ is not a } \mu\text{-SW-NE tableau}, \\ (-q^{-1})^{\ell(Q_\lambda)} q^{\gamma(Q_\lambda)} \prod_{i_p \in P_{Q\lambda}} e_{b_i(p)}(p), & \text{if } Q_\lambda \text{ is a } \mu\text{-SW-NE tableau,} \end{cases}$$

where $e_{b_i(p)}(p)$ is the $p$th column of the tableau $Q_\lambda$.
where $\gamma(Q_{\lambda})$ is the number of $j \geq j' + 1$ in $Q_{\lambda}$ with $j$ and $j + 1$ in the same row of $R_{\mu}$ and $\ell(Q_{\lambda})$ is the number of $j \leq j' + 1$ in $Q_{\lambda}$ with $j$ and $j + 1$ in the same row of $R_{\mu}$.

**Example 3.2.** Let $n = 24$, $r = 5$, $m_1 = m_2 = m_4 = m_5 = 24$, $m = 120$, and $\mu = ((5, 1), (3, 3, 1, 1), \emptyset, (2, 2, 2), (4))$. We will insert the up-down sequence of Example 2.7. First we apply the bijection (2.2) to give the variables their color superscript thereby converting

$I = [7, 11, 12, 12, 4][110][48, 70, 75][75, 75, 30][1][50][72, 25][16, 18][119, 97][5, 80, 79, 25]$

to

$[1^{(1)}, 2^{(1)}, 3^{(1)}, 4^{(1)}, 5^{(1)}][6^{(1)}][7^{(1)}, 8^{(1)}, 9^{(1)}][10^{(1)}, 11^{(1)}, 12^{(1)}][13^{(1)}][14^{(1)}][15^{(1)}, 16^{(1)}][17^{(1)}, 18^{(1)}][19^{(1)}, 20^{(1)}][21^{(1)}, 22^{(1)}, 23^{(1)}, 24^{(1)}]$

Upon inserting these variables we get

$Q_{\lambda} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \\ 13 & 14 & \emptyset, 15 & 16 & 21 & 22 & 23 & 24 \end{pmatrix}$

and

$P_{\lambda} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \\ 13 & 14 & \emptyset, 15 & 16 & 21 & 22 & 23 & 24 \end{pmatrix}$

The weight of $Q_{\lambda}$ is the same as the weight of the sequence $I$, shown in Example 2.3.

**Theorem 3.3.** Let $\mu \in \mathcal{P}_{n, r}$, then

$$q_{\mu}(x; q, u) = \sum_{\lambda \in \mathcal{P}_{n, r}} \left( \sum_{Q_{\lambda}} \text{wt}_{\mu}(Q_{\lambda}) \right) s_{\lambda}(x),$$

where the inner sum is over all standard tableaux $Q_{\lambda}$ of shape $\lambda$.

**Proof.** Comparing (2.14), (2.15) and (3.2), (3.3), we see that our insertion satisfies

if \( (P_{\lambda}, Q_{\lambda}) \stackrel{\text{RSK}}{\rightarrow} x_{i_1}, \ldots, x_{i_n} \), then \( \text{wt}_{\mu}(i_1, \ldots, i_n) = \text{wt}_{\mu}(Q_{\lambda}) \).

We now apply RSK insertion to the formula for $q_{\mu}$ found in Corollary 2.4:

$$q_{\mu}(x; q, u) = \sum_{i_1, \ldots, i_n} \text{wt}_{\mu}(i_1, \ldots, i_n) x_{i_1} \cdots x_{i_n}$$

$$= \sum_{\lambda \in \mathcal{P}_{n, r}} \sum_{Q_{\lambda}} \text{wt}_{\mu}(Q_{\lambda}) x_{P_{\lambda}}$$

$$= \sum_{\lambda \in \mathcal{P}_{n, r}} \sum_{Q_{\lambda}} \text{wt}_{\mu}(Q_{\lambda}) \sum_{P_{\lambda}} x_{P_{\lambda}}$$

$$= \sum_{\lambda \in \mathcal{P}_{n, r}} \sum_{Q_{\lambda}} \text{wt}_{\mu}(Q_{\lambda}) s_{\lambda}(x),$$
where $P'$ varies over all column-strict tableaux of shape $\lambda$ and $Q'$ varies over all standard tableaux of shape $\lambda$.

The Schur functions $s_\lambda$ are linearly independent [Mac], Appendix B (7.4), so comparing coefficients of $s_\lambda$ in (2.12) and Theorem 3.3 gives

**Corollary 3.4.** For $\lambda, \mu \in \mathcal{P}_{n,r}$, we have

$$
\chi_\lambda^\mu(a_\mu) = \sum_{Q} w^\mu(Q)\chi_\lambda(Q),
$$

where $\chi_\lambda^\mu(a_\mu)$ is the irreducible character of $H_{n,r}$ indexed by $\lambda$ and evaluated at $a_\mu$ and the sum is over all standard tableaux $Q$ of shape $\lambda$.

**Remark 3.5.** Upon setting $q = 1$ and $u_i = \zeta^i$, the formulas in Theorem 3.3 and Corollary 3.4 become a symmetric function identity (3.4) $p_\mu(x) = \sum_{\lambda \in \mathcal{P}_{n,r}} \left( \sum_{Q} w^\mu(Q)\chi_\lambda(Q) \right) s_\lambda(x)$, and a character formula (3.5) $\chi_\lambda^\mu(w_\mu) = \sum_{Q} w^\mu(Q)\chi_\lambda(Q)$, for the complex reflection group $G_{n,r}$.

**Remark 3.6.** Let $f_\lambda = \dim(V_\lambda^1) = \chi_\lambda^1(1)$. This dimension is equal to the number of standard tableaux of shape $\lambda$. As a special case of our insertion, we can restrict to sequences $x_{i_1}^{(k_1)}, \ldots, x_{i_n}^{(k_n)}$ where $i_1, \ldots, i_n$ is a permutation of $1, \ldots, n$ and $1 \leq k_i \leq r$. There are $n!r^n$ such sequences. Furthermore, when we insert these special sequences, we get a pair $(P, Q)$ of standard tableaux (the column-strict tableau $P$ is standard because all the subscripts are unique). Thus, our modified RSK insertion gives a bijective proof of the identity (3.6) $n!r^n = \sum_{\lambda \in \mathcal{P}_{n,r}} f_\lambda^1$, which also follows by decomposing the regular representation of $G_{n,r}$ into irreducibles and comparing dimensions.

**References**


DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, MACALESTER COLLEGE, SAINT PAUL, MINNESOTA 55105
*E-mail address: acantrell@macalester.edu*

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, MACALESTER COLLEGE, SAINT PAUL, MINNESOTA 55105
*E-mail address: halverson@macalester.edu*

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, MACALESTER COLLEGE, SAINT PAUL, MINNESOTA 55105
*E-mail address: btmiller@macalester.edu*