Tensor power multiplicities for symmetric and alternating groups and dimensions of irreducible modules for partition algebras

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Abstract

The partition algebra $P_k(n)$ and the symmetric group $S_n$ are in Schur-Weyl duality on the $k$-fold tensor power $M_n^{\otimes k}$ of the permutation module $M_n$ of $S_n$, so there is a surjection $P_k(n) \to Z_k(n) := \text{End}_{S_n}(M_n^{\otimes k})$, which is an isomorphism when $n \geq 2k$. We prove a dimension formula for the irreducible modules of the centralizer algebra $Z_k(n)$ in terms of Stirling numbers of the second kind. Via Schur-Weyl duality, these dimensions equal the multiplicities of the irreducible $S_n$-modules in $M_n^{\otimes k}$. Our dimension expressions hold for any $n \geq 1$ and $k \geq 0$. Our methods are based on an analog of Frobenius reciprocity that we show holds for the centralizer algebras of arbitrary finite groups and their subgroups acting on a finite-dimensional module. This enables us to generalize the above result to various analogs of the partition algebra including the centralizer algebra for the alternating group acting on $M_n^{\otimes k}$ and the quasi-partition algebra corresponding to tensor powers of the reflection representation of $S_n$.

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1 Introduction

The partition algebras $P_k(\xi), \xi \in \mathbb{C}$, were introduced by Martin ([M1], [M2], [M3]) to study the Potts lattice model of interacting spins in statistical mechanics. As shown by Jones [J], there is a Schur-Weyl duality between the partition algebra $P_k(n)$ and the symmetric group $S_n$ acting as centralizers of each other on the $k$-fold tensor power $M_n^{\otimes k}$ of the $n$-dimensional permutation module $M_n$ for $S_n$ over $\mathbb{C}$. The surjective algebra homomorphism given in [J] (see also [HR Thm. 3.6]),

$P_k(n) \to Z_k(n) := \text{End}_{S_n}(M_n^{\otimes k}) = \{ T \in \text{End}(M_n^{\otimes k}) | T(\sigma.u) = \sigma.T(u) \ \forall \sigma \in S_n, u \in M_n^{\otimes k} \}$,

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is an isomorphism when \( n \geq 2k \).

The partition algebra \( P_k(\xi) \) for \( k \geq 1 \) has a basis over \( \mathbb{C} \) indexed by set partitions of the set \( \{1, 2, \ldots, 2k\} \) into disjoint nonempty blocks. An example of such a set partition for \( k = 7 \) is \( \{1, 9, 11 \mid 2, 13 \mid 3 \mid 4, 7, 8 \mid 5, 6, 12, 14\} \), which has 5 blocks. The Stirling number \( \{\begin{array}{c} 2k \\ r \end{array}\} \) counts the number of ways to partition \( 2k \) objects into \( r \) nonempty disjoint blocks, so it follows that

\[
\dim P_k(\xi) = \sum_{r=1}^{2k} \left\{ \begin{array}{c} 2k \\ r \end{array}\right\} = B(2k), \quad \text{(the (2k)th Bell number)}.\]

In \( P_{k+1}(\xi) \), the basis elements indexed by set partitions which have \( k + 1 \) and \( 2(k + 1) \) in the same block form a subalgebra \( P_{k+1}(\xi) \) with \( \dim P_{k+1}(\xi) = B(2k + 1) \). If we regard \( M_n \) as a module for the symmetric group \( S_{n-1} \) by restriction, there is a surjective algebra homomorphism \( P_{k+\frac{1}{2}}(n) \to Z_{k+\frac{1}{2}}(n) := \text{End}_{S_{n-1}}(M_n^{\otimes k}) \), which is an isomorphism if \( n \geq 2k \). These intermediate algebras play an important role in understanding the structure and representation theory of partition algebras (see for example, \([MR, HR]\)), and they are a crucial component of the work in this paper.

The irreducible modules for \( P_k(n) \) and \( P_{k+\frac{1}{2}}(n) \) are labeled by partitions \( \nu \) of \( n \). Schur-Weyl duality implies that the irreducible modules for \( Z_k(n) \) are also indexed by partitions \( \lambda \) of \( n \), and for \( Z_{k+\frac{1}{2}}(n) \), by partitions \( \mu \) of \( n - 1 \). The modules \( S^\lambda_n \) (resp. \( S^\mu_{n-1} \)) occurring in \( M_n^{\otimes k} \) are indexed by partitions with the property that the partition \( \nu = \lambda^\# \) (resp. \( \nu = \mu^\# \)) that results from deleting the largest part of \( \lambda \) (resp. of \( \mu \)) must satisfy \( 0 \leq |\nu| \leq k \), where \(|\nu|\) is the sum of the parts of \( \nu \).

In this paper, we

- establish general restriction/induction results for centralizer algebras, proving in Theorem 2.7 that an analog of Frobenius reciprocity for groups holds for their centralizer algebras;

- give restriction/induction Bratteli diagrams for the symmetric group-subgroup pair \((S_n, S_{n-1})\) and for the alternating group-subgroup pair \((A_n, A_{n-1})\);

- use the reciprocity results to determine expressions for the dimensions of the irreducible modules for \( Z_k(n) \), and \( Z_{k+\frac{1}{2}}(n) \) (in Theorem 5.13(a) and (b)), and for \( P_k(\xi) \), \( P_{k+\frac{1}{2}}(\xi) \) (in Corollary 5.26);

- determine the dimensions of the centralizer algebras \( Z_k(n) \) and \( Z_{k+\frac{1}{2}}(n) \) (in Theorem 5.13(c) and (d));

- apply the restriction/induction results to the pair \((S_n, A_n)\) (resp. \((S_{n-1}, A_{n-1})\)) to determine the dimensions of the irreducible modules for the centralizer algebras \( \tilde{Z}_k(n) := \text{End}_{A_n}(M_n^{\otimes k}) \) and \( \tilde{Z}_{k+\frac{1}{2}}(n) := \text{End}_{A_{n-1}}(M_n^{\otimes k}) \) (in Theorem 6.1(a) and (b));

- determine dimension formulas for the centralizer algebras \( \tilde{Z}_k(n) \) and \( \tilde{Z}_{k+\frac{1}{2}}(n) \) (in Theorem 6.1);

- compute the dimensions of the irreducible modules for the centralizer algebras \( QZ_k(n) := \text{End}_{S_n}(R_n^{\otimes k}) \), where \( R_n := S_{n-1}^{[n-1,1]} \) is the \((n-1)\)-dimensional irreducible reflection module of
$S_n$ corresponding to the partition $[n - 1, 1]$ of $n$, and for their relatives $QZ_{k+\frac{1}{2}}(n)$, $\widehat{QZ}_k(n)$, and $\overline{QZ}_{k+\frac{1}{2}}(n)$ (in Theorem 7.1) and give Bratteli diagrams corresponding to $R_n$ for the group-subgroup pairs $(S_n, S_{n-1})$ and $(A_n, A_{n-1})$.

By Schur-Weyl duality, the dimension of an irreducible module for the centralizer algebra equals the multiplicity of the corresponding irreducible module for the group. Consequently, our dimension formulas also

- determine the multiplicities of irreducible modules for $S_n, S_{n-1}, A_n,$ and $A_{n-1}$ in $M_n^{\otimes k}$ and in $FP_{n}^{\otimes k}$ for all $n, k \in \mathbb{Z}_{>0}$ (in Theorems 5.14(a), 6.1, and 7.1).

In Definition 5.5 we introduce certain sequences of partitions,

$$\pi = \{[\lambda_1, 0, \ldots, 0] = \pi^{(0)} \subseteq \pi^{(1)} \subseteq \pi^{(2)} \subseteq \ldots \subseteq \pi^{(r)} = \lambda\},$$

(termed $r$-sequences) corresponding to the partition $\lambda = [\lambda_1, \ldots, \lambda_n]$ ($\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$) of $n$ with the property that the partitions $\pi^{(j)} = [\pi_1^{(j)}, \ldots, \pi_n^{(j)}]$ in the sequence satisfy

$$\pi_1^{(1)} = \pi_2^{(1)} = \pi_2^{(2)} = \pi_3^{(2)} = \ldots = \pi_{r-1}^{(r-1)} = \pi_r^{(r-1)} = \lambda_r.$$

The dimension expressions we obtain involve binomial coefficients, Stirling numbers of the second kind, and the integers $F^\lambda = \sum_\pi f^\lambda_\pi$, where the sum is over all $r$-sequences $\pi$ for $\lambda$, and $f^\lambda_\pi$ equals the number of standard Young tableaux of shape $\lambda$ such that $\pi^{(j)}$ contains the numbers $1, \ldots, |\pi^{(j)}|$ for each $0 \leq j \leq r$. The dimensions of the centralizer algebras $Z_k(n)$ and $\hat{Z}_k(n)$ were determined previously and can be found in [HR] and [B1, B2], respectively. In this work, they are direct consequences of the dimension formulas for the irreducible modules. This is a general phenomenon: If $Z_k(G) := \text{End}_G(X^{\otimes k})$ for a self-dual module $X$ of a group $G$, then $\dim Z_k(G) = \dim (X^{\otimes 2k})^G$, where $(X^{\otimes 2k})^G$ is the space of $G$-invariants in $X^{\otimes 2k}$. Therefore, $\dim Z_k(G)$ is the multiplicity of the trivial $G$-module $G^\ast$ in $X^{\otimes 2k}$; equivalently, by Schur-Weyl duality, it is the dimension of the irreducible module associated to $G^\ast$ for the centralizer algebra $Z_{2k}(G)$ (see Section 2 for details).

When $n \geq 2k$, the irreducible module for the partition algebra $P_k(n)$ corresponding to the partition $\lambda$ of $n$ has dimension equal to the number of paths $(P, T)$, where $P$ is a set partition of $\{1, \ldots, k\}$ and $T$ is a standard Young tableau of shape $\lambda^\#$ whose entries depend on $P$. As we discuss in Remark 5.27, these pairs were studied previously in [CDDS], and the number of such pairs equals the number of paths in the Bratteli diagram for the partition algebra from the root at the top to $\lambda$ at level $k$.

Motivated by the work of Goupil and Chauve [GC] on Kronecker tableaux and Kronecker coefficients, Daugherty and Orellana in [DO] introduced the quasi-partition algebras $QP_k(\xi), \xi \in \mathbb{C}$, and showed that there is a surjection $QP_k(n) \twoheadrightarrow QZ_k(n) = \text{End}_{S_n}(R_n^{\otimes k})$ for $R_n = S_n^{[n-1,1]}$, which is an isomorphism when $n \geq 2k$. The dimensions for the irreducible modules for $QP_k(\xi)$, with $\xi$ generic, are the same as the dimensions for $n \geq 2k$, and so are given by the dimension formulas in Section 7 below. These expressions differ from the ones that appear in [DO], which were based on results in [GC] and hold whenever $n \geq 2k$, as the ones in Section 7 are valid for all $k$ and $n$.

Using exponential generating functions from [GC], Ding [D] derived a formula for the multiplicity of the irreducible $S_n$-module $S_\lambda^k$ indexed by the partition $\lambda = [\lambda_1, \ldots, \lambda_n]$ in $M_n^{\otimes k}$ when $1 \leq k \leq n - \lambda_2$ and used that to obtain an expression for the multiplicity of $S_\lambda^k$ in tensor powers.
$R_n^k$ of its reflection module $R_n = S_n^{[n-1,1]}$. The first is a special case of part (a) of Theorem 5.13 below and the second a special case of Theorem 7.1. As shown in [D, Sec. 3], when $1 \leq k \leq n - \lambda_2$, these multiplicity formulas can be used to bound the mixing time of a Markov chain on $S_n$.

### 2 Restriction/Induction and Dimensions

We begin with some general results on restriction and induction for centralizer algebras and then apply these results to the group-subgroup pairs $(S_n, S_{n-1})$, $(S_n, A_n)$, and $(A_n, A_{n-1})$ acting on the $k$-fold tensor power of the $n$-dimensional permutation module $M_n$. This will enable us to determine the dimension of the centralizer algebras and their irreducible modules.

Suppose $G$ is a finite group and $H$ is a subgroup of $G$. Assume $\{G^\lambda\}_{\lambda \in \Lambda_G}$ and $\{H^\alpha\}_{\alpha \in \Lambda_H}$ are the corresponding sets of irreducible modules for these groups over $C$. We suppose that the restriction from $G$ to $H$ on $G^\lambda$ is given by

$$\text{Res}_H^G(G^\lambda) = \bigoplus_{\alpha \in \Lambda_H} c_{\lambda\alpha}^H H^\alpha. \quad (2.1)$$

Then by Frobenius reciprocity, induction from $H$ to $G$ is given by

$$\text{Ind}_H^G(H^\alpha) = \bigoplus_{\lambda \in \Lambda_G} c_{\lambda\alpha}^G G^\lambda. \quad (2.2)$$

Assume now that $X$ is a finite-dimensional $G$-module, and consider the centralizer algebra $Z_X(G) = \text{End}_G(X) = \{T \in \text{End}(X) \mid T(g.x) = g.T(x), \forall g \in G, x \in X\}$. Regarding $X$ as a module for the subgroup $H$ of $G$ by restriction, we have reverse inclusion of the centralizer algebras $Z_X(G) \subseteq Z_X(H) = \text{End}_H(X)$. Let $\Lambda_{X,G}$ (resp. $\Lambda_{X,H}$) denote the subset of $\Lambda_G$ (resp. of $\Lambda_H$) corresponding to the irreducible $G$-modules (resp. $H$-modules) which occur in $X$ with multiplicity at least one. Then Schur-Weyl duality implies the following:

- the irreducible $Z_X(G)$-modules $Z_{X,G}^\lambda$ are in bijection with the elements of $\lambda \in \Lambda_{X,G}$;
- the decomposition of $X$ into irreducible $G$-modules is given by

$$X \cong \bigoplus_{\lambda \in \Lambda_{X,G}} d_{X,G}^\lambda G^\lambda, \quad (2.3)$$

where $d_{X,G}^\lambda = \dim Z_{X,G}^\lambda$;
- the decomposition of $X$ into irreducible $Z_X(G)$-modules is given by

$$X \cong \bigoplus_{\lambda \in \Lambda_{X,G}} d_{G^\lambda} Z_{X,G}^\lambda, \quad (2.4)$$

where $d_{G^\lambda} = \dim G^\lambda$;
- as a bimodule for $G \times Z_X(G)$,

$$X \cong \bigoplus_{\lambda \in \Lambda_{X,G}} \left( G^\lambda \otimes Z_{X,G}^\lambda \right); \quad (2.5)$$
\[ \text{dim } Z_X(G) = \sum_{\lambda \in \Lambda_{X,G}} (\text{dim } Z_{X,G}^\lambda)^2 = \sum_{\lambda \in \Lambda_{X,G}} (d_{X,G}^\lambda)^2. \tag{2.6} \]

There is a corresponding Frobenius reciprocity for centralizer algebras of the group-subgroup pair \((G, H)\), as indicated in the next result.

**Theorem 2.7.** For a finite-dimensional \(G\)-module \(X\), let \(Z_X(G) = \text{End}_G(X)\) and \(Z_X(H) = \text{End}_H(X)\). Let \(\Lambda_{X,G}\) (resp. \(\Lambda_{Y,H}\)) be the set of indices \(\lambda \in \Lambda_G\) (resp. \(\alpha \in \Lambda_H\)) such that \(G^\lambda\) (resp. \(H^\alpha\)) occurs in \(X\) with multiplicity \(\geq 1\), and let \(Z_{X,G}^\lambda\) (resp. \(Z_{Y,H}^\alpha\)) denote the corresponding irreducible \(Z_X(G)\)-module (resp. \(Z_X(H)\)-module). Assume \(c_{\alpha}^\lambda\) is as in (2.1) and (2.2) above. Then the following hold:

(a) \(\text{Res}_{Z_X(G)}^G(Z_{X,H}^\alpha) = \bigoplus_{\lambda \in \Lambda_{X,G}} c_{\alpha}^\lambda Z_{X,G}^\lambda.\)

(b) \(\text{For } \alpha \in \Lambda_{X,H}, \ d_{X,H}^\alpha = \sum_{\lambda \in \Lambda_{X,G}} c_{\alpha}^\lambda d_{X,G}^\lambda, \ \text{where } d_{X,H}^\alpha := \dim Z_{X,H}^\alpha \text{ and } d_{X,G}^\lambda := \dim Z_{X,G}^\lambda.\)

(c) \(\text{Ind}_{Z_X(G)}^H(Z_{X,G}^\lambda) := Z_X(H) \otimes_{Z_X(G)} Z_{X,G}^\lambda = \bigoplus_{\alpha \in \Lambda_{X,H}} c_{\alpha}^\lambda Z_{X,H}^\alpha.\)

(d) As a \(Z_X(G)\)-module (via multiplication on the left),

\[ Z_X(H) = \bigoplus_{\lambda \in \Lambda_{X,G}} \left( \sum_{\alpha \in \Lambda_{X,H}} c_{\alpha}^\lambda d_{X,H}^\alpha \right) Z_{X,G}^\lambda. \]

(e) Assume \(Y\) is an \(H\)-module and set \(X = \text{Ind}_H^G(Y)\). Let \(Z_Y(H) = \text{End}_H(Y)\), and let \(Z_{Y,H}^\alpha\), \(\alpha \in \Lambda_{Y,H}\), be the irreducible \(Z_Y(H)\)-modules. Then for \(\lambda \in \Lambda_{X,G}\),

\[ \dim Z_{X,G}^\lambda = \sum_{\alpha \in \Lambda_{Y,H}} c_{\alpha}^\lambda \dim Z_{Y,H}^\alpha. \]

**Proof.** (a) and (b): By Schur-Weyl duality, \(X \cong \bigoplus_{\lambda \in \Lambda_{X,G}} (G^\lambda \otimes Z_{X,G}^\lambda)\) as a \((G \times Z_X(G))\)-bimodule. Therefore, as an \((H \times Z_X(G))\)-bimodule,

\[ X \cong \sum_{\lambda \in \Lambda_{X,G}} \left( \sum_{\alpha \in \Lambda_{H}} c_{\alpha}^\lambda H^\alpha \right) \otimes Z_{X,G}^\lambda \cong \sum_{\alpha \in \Lambda_{H}} H^\alpha \otimes \left( \sum_{\lambda \in \Lambda_{X,G}} c_{\alpha}^\lambda Z_{X,G}^\lambda \right). \tag{2.8} \]

This says that the \(H\)-module \(H^\alpha\) occurs as a summand of \(X\) with multiplicity equal to \(\sum_{\lambda \in \Lambda_{X,G}} c_{\alpha}^\lambda \dim Z_{X,G}^\lambda\). But from the decomposition

\[ X \cong \bigoplus_{\alpha \in \Lambda_{X,H}} (H^\alpha \otimes Z_{X,H}^\alpha), \tag{2.9} \]
we know that the only $H$-summands occurring in $X$ are those with $\alpha \in \Lambda_{X,H}$, and $H^\alpha$ has multiplicity $\dim Z^\alpha_{X,H}$ in $X$. Therefore, the sum in (2.3) must be over $\alpha \in \Lambda_{X,H}$, and we have $\dim Z^\alpha_{X,H} = \sum_{\lambda \in \Lambda_{X,G}} c^\alpha_{\lambda} \dim Z^\lambda_{X,G}$, as claimed in (b). Moreover, the restriction of the $(H \times Z_X(H))$-decomposition of $X$ in (2.9) to $H \times Z_X(H)$ gives $X \cong \bigoplus_{\alpha \in \Lambda_{X,H}} H^\alpha \otimes \operatorname{Res} Z_{X(G)}(H^\alpha) \left( Z_{X,H}^\alpha \right)$. Since the decomposition of $X$ as a $(H \times Z_X(G))$-bimodule is unique, $\operatorname{Res} Z_{X(G)}(H^\alpha) \left( Z_{X,H}^\alpha \right) = \bigoplus_{\lambda \in \Lambda_{X,G}} c^\lambda_{\alpha} Z^\lambda_{X,G}$ must hold, as asserted in (a). Note that part (b) is just the dimension version of this relation.

For part (c), we use the following standard result. Assume $A$ is an algebra and $B$ is a subalgebra of $A$. Let $W$ be an $A$-module and $V$ be a $B$-module. Then

$$\operatorname{Hom}_A(A \otimes_B V, W) = \operatorname{Hom}_B(V, \operatorname{Res}^A_B(W)). \quad (2.10)$$

Now suppose $A = Z_X(H)$, $B = Z_X(G)$, $V = Z^\lambda_{X,G}$, and $W = Z^\alpha_{X,H}$. Then

$$\operatorname{Hom}_A(\operatorname{Ind}^A_B(Z^\lambda_{X,G}), Z^\alpha_{X,H}) = \operatorname{Hom}_B(Z^\lambda_{X,G}, \operatorname{Res}^A_B(Z^\alpha_{X,H})) = \operatorname{Hom}_B \left( Z^\lambda_{X,G}, \bigoplus_{\mu \in \Lambda_{X,G}} c^\mu_{\alpha} Z^\mu_{X,G} \right).$$

Taking dimensions on both sides shows that $\dim \operatorname{Hom}_A(\operatorname{Ind}^A_B(Z^\lambda_{X,G}), Z^\alpha_{X,H}) = c^\lambda_{\alpha}$, and thus,

$$\operatorname{Ind}^B_Z(\lambda_{X,G}) = \bigoplus_{\alpha \in \Lambda_{X,H}} c^\lambda_{\alpha} Z^\alpha_{X,H}. \quad (d)$$

Since $Z_X(H)$ is a semisimple algebra, Wedderburn theory tells us $Z_X(H) = \bigoplus_{\alpha \in \Lambda_{X,H}} d^\alpha_{X,H} Z^\alpha_{X,H}$, where $d^\alpha_{X,H} = \dim Z^\alpha_{X,H}$. Restricting to $Z_X(G)$ gives

$$\operatorname{Res} Z_{X(G)}(H^\alpha) \left( Z_X(H) \right) = \bigoplus_{\alpha \in \Lambda_{X,H}} d^\alpha_{X,H} \operatorname{Res} Z_{X(G)}(H^\alpha) \left( Z_{X,H}^\alpha \right) = \bigoplus_{\alpha \in \Lambda_{X,H}} \left( \bigoplus_{\lambda \in \Lambda_{X,G}} c^\lambda_{\alpha} d^\alpha_{X,H} \right) Z^\lambda_{X,G},$$

by part (a).

(c) The proof here is similar in spirit to that in parts (a) and (b). With $Y$ an $H$-module, suppose $Y = \bigoplus_{\alpha \in \Lambda_{Y,H}} y^\alpha H^\alpha$, and assume $X := \operatorname{Ind}^G_H(Y) = \bigoplus_{\alpha \in \Lambda_{X,G}} x^\alpha G^\alpha$. Then

$$X = \operatorname{Ind}^G_H(Y) = \sum_{\alpha \in \Lambda_{Y,H}} y^\alpha \operatorname{Ind}^G_H(H^\alpha) = \sum_{\alpha \in \Lambda_{Y,H}} y^\alpha \left( \sum_{\lambda \in \Lambda_G} c^\lambda_{\alpha} G^\lambda \right) = \sum_{\lambda \in \Lambda_G} \left( \sum_{\alpha \in \Lambda_{Y,H}} c^\lambda_{\alpha} y^\alpha \right) G^\lambda,$$

so that the sum must be over $\lambda \in \Lambda_{X,G}$, and

$$\dim Z^\lambda_{X,G} = x^\lambda = \sum_{\alpha \in \Lambda_{Y,H}} c^\lambda_{\alpha} y^\alpha = \sum_{\alpha \in \Lambda_{Y,H}} c^\lambda_{\alpha} \dim Z^\alpha_{Y,H}$$

for all $\lambda \in \Lambda_{X,G}$.

The following proposition will be used in Section 7 to relate multiplicities in the tensor power of the reflection module of the symmetric group to multiplicities in tensor powers of the permutation module. Assume $G$ is a finite group and $W$ is a $G$-module over $\mathbb{C}$. Let $V = G^* \oplus W$ be the extension of $W$ by the trivial $G$-module $G^*$. Define $Z_k(G) = \operatorname{End}_G(V \otimes^k)$ and $QZ_k(G) = \operatorname{End}_G(W \otimes^k)$, and let $\Lambda_{k,G} \subseteq \Lambda(G)$ (resp., $q\Lambda_{k,G} \subseteq \Lambda(G)$) index the irreducible $G$-modules that appear in $V \otimes^k$ (resp., in $W \otimes^k$) with multiplicity at least one. Let $Z^\lambda_k$ (resp., $QZ^\lambda_k$) denote the irreducible $Z_k(G)$-module (resp., $QZ_k(G)$-module) indexed by $\lambda \in \Lambda_{k,G}$ (resp., $\lambda \in q\Lambda_{k,G}$).

\[ \blacksquare \]
Proposition 2.11. With notation as in the previous paragraph,

(a) \( \dim \mathbb{Q}Z^\lambda_k = \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} \dim Z^\lambda_\ell \),

(b) If \( W \) is a self-dual \( G \)-module, then \( \dim \mathbb{Q}Z_k(G) = \dim \mathbb{Q}Z^\bullet_{2k} = \sum_{\ell=0}^{2k} (-1)^{2k-\ell} \binom{2k}{\ell} \dim Z^\bullet_\ell \),

where \( \mathbb{Q}Z^\bullet_{2k} \) is the irreducible \( \mathbb{Q}Z_k(G) \)-module corresponding to \( G^\bullet \); equivalently, the space of \( G \)-invariants in \( W \otimes 2^k \).

Proof. Let \( \chi_V, \chi_\bullet, \chi_W \) denote the characters of \( V, G^\bullet, \) and \( W \), respectively, so that \( \chi_V = \chi_\bullet + \chi_W \).

Then \( \chi_V \otimes k = (\chi_\bullet + \chi_W)^k = \sum_{\ell=0}^{k} \binom{k}{\ell} \chi_\ell \), and the binomial inverse of this statement is

\[
\chi_W \otimes k = \chi_\ell = \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} \chi_\ell = \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} \chi_V \otimes \ell. \tag{2.12}
\]

By Schur-Weyl duality (2.3) we have

\[
W \otimes k = \bigoplus_{\lambda \in \Lambda_k} qd^\lambda_k \mathcal{G}^\lambda, \quad \text{where} \quad qd^\lambda_k = \dim \mathbb{Q}Z^\lambda_k. \tag{2.13}
\]

Computing the character of (2.13) and equating it with (2.12) gives

\[
\chi_W \otimes k = \chi_\ell = \sum_{\lambda \in \Lambda_k} qd^\lambda_k \chi_\lambda = \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} \left( \sum_{\lambda \in \Lambda_k} qd^\lambda_\ell \chi_\lambda \right),
\]

where \( d^\lambda_k = \dim Z^\lambda_k \), and \( \chi_\lambda \) is the character of \( G^\lambda \). Equating the coefficient of \( \chi_\lambda \) (working in the ring of class functions on \( G \)) gives part (a). Since \( W \) is isomorphic to its dual as a \( G \)-module, part (b) is the special case of part (a) with \( \lambda = \bullet \) (the index of the trivial module):

\[
\dim \mathbb{Q}Z_k(G) = \dim \mathbb{Q}Z^\bullet_{2k} = \sum_{\ell=0}^{2k} (-1)^{2k-\ell} \binom{2k}{\ell} \dim Z^\bullet_\ell. \quad \square
\]

3 Irreducible modules for symmetric and alternating groups and their centralizer algebras

The irreducible \( S_n \)-modules are labeled by partitions of \( n \), so that \( \Lambda_{S_n} = \{ \lambda | \lambda \vdash n \} \). When writing \( \lambda = [\lambda_1, \ldots, \lambda_n] \vdash n \), we always assume that the parts of the partition \( \lambda \) are arranged so that \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0 \), and \( |\lambda| = n \) (the sum of the parts). We identify a partition with its Young diagram, so for \( \lambda = [6, 4, 3, 2^2] \vdash 17 \), we have

\[
\lambda = \begin{array}{cccccccc}
\text{Young Diagram}
\end{array}
\]
The hook length \( h(b) \) of a box \( b \) in the diagram is 1 plus the number of boxes below \( b \) in the same column plus the number of boxes to the right of \( b \) in the same row, and \( h(b) = 1 + 3 + 2 = 6 \) for the shaded box above. The dimension of the irreducible \( S_n \)-module \( S^\lambda_n \), which we denote \( f^\lambda \), can be easily computed by the well-known hook-length formula

\[
f^\lambda = \frac{n!}{\prod_{b \in \lambda} h(b)}, \tag{3.1}
\]

where the denominator is the product of the hook lengths as \( b \) ranges over the boxes in the Young diagram of \( \lambda \). This is equal to the number of standard Young tableaux of shape \( \lambda \), where a standard Young tableau \( T \) is a filling of the boxes in the Young diagram of \( \lambda \) with the numbers \( \{1, \ldots, n\} \) such that the entries increase in every row from left to right and in every column from top to bottom.

The restriction and induction rules for irreducible symmetric group modules \( S^\lambda_n \) are well known (see for example [JK, Thm. 2.43]):

\[
\text{Res}_{S_{n-1}}^{S_n}(S^\lambda_n) = \bigoplus_{\mu = \lambda - \Box} S^\mu_{n-1} \quad \text{Ind}_{S_{n+1}}^{S_n}(S^\lambda_n) = \bigoplus_{\kappa = \lambda + \Box} S^\kappa_{n+1}, \tag{3.2}
\]

where the first sum is over all partitions \( \mu \) of \( n - 1 \) obtained from \( \lambda \) by removing a box from the end of a row of the diagram of \( \lambda \), and the second sum is over all partitions \( \kappa \) of \( n + 1 \) obtained by adding a box to the end of a row of \( \lambda \).

For \( \lambda \vdash n \), let \( \lambda^* \) be the conjugate (transpose) partition. Since \( S^\lambda_n \cong S_n^{[n]} \otimes S_n^\lambda \), where \( S_n^{[n]} \) is the one-dimensional irreducible \( S_n \)-module indexed by the partition of \( n \) into \( n \) parts of size one, which is the sign representation, \( S^\lambda_n \cong S_n^\lambda \) as \( A_n \)-modules. Thus, we may assume that \( \lambda \geq \lambda^* \) (in the dominance order), which is to say that for the first part where \( \lambda \) and \( \lambda^* \) differ \( \lambda_j > \lambda_j^* \). Then by Clifford theory, it is known that

\[
\text{Res}_{A_n}^{S_n}(S^\lambda_n) = \begin{cases} A^\lambda_n \cong \text{Res}_{A_n}^{S_n}(S^\lambda^*_n) & \text{if } \lambda \neq \lambda^*, \\ A^\lambda_n + A^{\lambda^*}_n & \text{if } \lambda = \lambda^*. \end{cases} \tag{3.3}
\]

where in the first case, \( A^\lambda_n \) is irreducible as an \( A_n \)-module; while in the second case, \( S^\lambda_n \) decomposes into the direct sum of two irreducible \( A_n \)-modules, \( S^\lambda_n = A^\lambda_n + A^{\lambda^*}_n \), such that \( \dim A^\lambda_n = \dim A^{\lambda^*}_n = \frac{1}{2} \dim S^\lambda_n = \frac{1}{2} f^\lambda \). Moreover,

\[
\Lambda_n = \{ \lambda \mid \lambda \vdash n, \lambda > \lambda^* \} \cup \{ \lambda^\pm \mid \lambda \vdash n, \lambda = \lambda^* \}.
\]

The restriction rules for alternating groups are the following (see [R, Thm. 6.1], or [Mb] which surveys how to derive these rules using Mackey’s theorem and Clifford theory):

\[
\text{Res}_{A_{n-1}}^{A_n}(A^\mu_n) = \bigoplus_{\mu = \lambda - \Box, \mu > \mu^*} A^\mu_{n-1} \oplus \left( \bigoplus_{\mu = \lambda - \Box, \mu = \mu^*} (A^\mu_{n-1} \oplus A^{\mu^*}_{n-1}) \right) \quad \text{if } \lambda > \lambda^*, \tag{3.4}
\]

\[
\text{Res}_{A_{n-1}}^{A_n}(A^{\lambda^\pm}_n) = \bigoplus_{\mu = \lambda - \Box, \mu > \mu^*} A^\mu_{n-1} \oplus \left( \bigoplus_{\mu = \lambda - \Box, \mu = \mu^*} A^{\lambda^\pm}_{n-1} \right) \quad \text{if } \lambda = \lambda^*.
\]
Now let $M_n$ be the $n$-dimensional permutation module for $S_n$, and set $X = M_n^\otimes k$ for $k \geq 0$ in applying the results of Section 2 where $M_n^\otimes 0 = S_n^{[n]}$ (the trivial $S_n$-module). Since $M_n$ will be fixed throughout, it convenient here to adopt the shorthand notation in Table 1. For all $k \in \mathbb{Z}_{\geq 0}$, $\Lambda_{k,S_n}$ (resp. $\Lambda_{k,A_n}$) is the set of indices for the irreducible $Z_k(n)$-summands (resp. $\tilde{Z}_k(n)$-summands) in $M_n^\otimes k$ with multiplicity at least one; similarly $\Lambda_{[k,S_n-1]}$ (resp. $\Lambda_{[k,A_n-1]}$) is the set of indices for the irreducible $Z_{k+\frac{1}{2}}(n)$-summands (resp. $\tilde{Z}_{k+\frac{1}{2}}(n)$-summands) in $M_n^\otimes k$ with multiplicity at least one.

<table>
<thead>
<tr>
<th>centralizer algebra</th>
<th>irreducible modules</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_k(n) := \text{End}_{S_n}(M_n^\otimes k)$</td>
<td>$Z^\lambda_{k,n}$, $\lambda \vdash n$, $\lambda \in \Lambda_{k,S_n} \subseteq \Lambda_{S_n}$</td>
</tr>
<tr>
<td>$Z_{k+\frac{1}{2}}(n) := \text{End}<em>{S</em>{n-1}}(M_n^\otimes k)$</td>
<td>$Z^\mu_{k+\frac{1}{2},n}$, $\mu \vdash n - 1$, $\mu \in \Lambda_{k,S_{n-1}} \subseteq \Lambda_{S_{n-1}}$</td>
</tr>
<tr>
<td>$\tilde{Z}<em>k(n) := \text{End}</em>{A_n}(M_n^\otimes k)$</td>
<td>$\tilde{Z}^\lambda_{k,n}$, $\lambda \vdash n$, $\lambda &gt; \lambda^*$, $\lambda \in \Lambda_{k,A_n} \subseteq \Lambda_{A_n}$</td>
</tr>
<tr>
<td>$\tilde{Z}<em>{k+\frac{1}{2}}(n) := \text{End}</em>{A_{n-1}}(M_n^\otimes k)$</td>
<td>$\tilde{Z}^\mu_{k+\frac{1}{2},n}$, $\mu \vdash n - 1$, $\mu &gt; \mu^*$, $\mu \in \Lambda_{k+\frac{1}{2},A_{n-1}} \subseteq \Lambda_{A_{n-1}}$</td>
</tr>
</tbody>
</table>

Table 1: Notation for the centralizer algebras and modules associated with the tensor product $M_n^\otimes k$ of the permutation module $M_n \cong S_n^{[n]} \oplus S_n^{[n-1,1]}$ of $S_n$ and its restriction to $S_{n-1}$, $A_n$, and $A_{n-1}$.

Theorem 2.7(b) together with (3.3) imply the following:

**Proposition 3.5.** Assume $\lambda \vdash n$, $\lambda \in \Lambda_{k,A_n}$, and $\lambda \geq \lambda^*$. Then

\[
\begin{align*}
\dim \tilde{Z}^\lambda_{k,n} &= \dim Z^\lambda_{k,n} + \dim Z^\lambda_{\mu,n}^\ast, & \text{if } \lambda > \lambda^*, \\
\dim \tilde{Z}^\lambda_{k,n}^\ast &= \dim \tilde{Z}^\lambda_{k,n} = \dim Z^\lambda_{k,n}, & \text{if } \lambda = \lambda^*.
\end{align*}
\]

(3.6)

**Example 3.7.** For $S_4$, we have $M_4^\otimes 3 = 5S_4^{[4]} \oplus 10S_4^{[3,1]} \oplus 5S_4^{[2]} \oplus 6S_4^{[2,1]} \oplus S_4^{[1]}$, and for $A_4$, $M_4^\otimes 3 = 6A_4^{[4]} \oplus 16A_4^{[3,1]} \oplus 5A_4^{[2]} \oplus 5A_4^{[1]}$, as can be seen in row $\ell = 3$ of Figures 1 and 2 where

\[
\begin{align*}
\dim \tilde{Z}_{3,4}^{[4]} &= \dim Z_{3,4}^{[4]} + \dim Z_{3,4}^{[14]} = 5 + 1 = 6, \\
\dim \tilde{Z}_{3,4}^{[3,1]} &= \dim Z_{3,4}^{[3,1]} + \dim Z_{3,4}^{[2]} = 10 + 6 = 16, \\
\dim \tilde{Z}_{3,4}^{[2]} &= \dim Z_{3,4}^{[2]} = 5.
\end{align*}
\]

4 Bratteli diagrams

Let $(G, H)$ be a pair consisting of a finite group $G$ and a subgroup $H \subseteq G$. As in Section 2 let $\{G^\lambda\}_{\lambda \in \Lambda_G}$ and $\{H^\alpha\}_{\alpha \in \Lambda_H}$ be the irreducible modules of $G$ and $H$ over $\mathbb{C}$ with restriction and induc-
By induction, we have the following isomorphisms for all $k$

$$\text{Res}_G^H(G^\lambda) = \bigoplus_{\alpha \in \Lambda_H} c^\lambda_\alpha H^\alpha \quad \text{and} \quad \text{Ind}_H^G(H^\alpha) = \bigoplus_{\lambda \in \Lambda_G} c^\lambda_\alpha G^\lambda. \quad (4.1)$$

Let $U^0 = G^*$, the trivial $G$-module, and assume for $k \in \mathbb{Z}_{\geq 0}$ that the $G$-module $U^k$ has been defined. Let $U^{k+\frac{1}{2}}$ be the $H$-module defined by $U^{k+\frac{1}{2}} := \text{Res}_H^G(U^k)$, and then let $U^{k+1}$ be the $G$-module specified by $U^{k+1} := \text{Ind}_H^G(U^{k+\frac{1}{2}})$. In this way, $U^\ell$ is defined inductively for all $\ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, and $U^k = (\text{Ind}_H^G(\text{Res}_H^G))^k(U^0)$ for all $k \in \mathbb{Z}_{\geq 0}$. The module $V := \text{Ind}_H^G(\text{Res}_H^G(U^0)) = U^1$ is isomorphic to $G/H$ as a $G$-module, where $G$ acts on the left cosets of $G/H$ by multiplication.

For a $G$-module $X$ and an $H$-module $Y$, the “tensor identity” says that $\text{Ind}_H^G(\text{Res}_H^G(X \otimes Y)) \cong X \otimes \text{Ind}_H^G(Y)$ (see for example [HR, (3.18)] for an explicit isomorphism). Hence, when $X = U^k$ and $Y = \text{Res}_H^G(U^0)$, this gives

$$\text{Ind}_H^G(\text{Res}_H^G(U^k)) \cong \text{Ind}_H^G(\text{Res}_H^G(U^k) \otimes \text{Res}_H^G(U^0)) \cong U^k \otimes \text{Ind}_H^G(\text{Res}_H^G(U^0)) = U^k \otimes V. \quad (4.2)$$

By induction, we have the following isomorphisms for all $k \in \mathbb{Z}_{\geq 0}$:

$$V^{\otimes k} \cong U^k \quad \text{(as G-modules)} \quad \text{and} \quad \text{Res}_H^G(V^{\otimes k}) \cong U^{k+\frac{1}{2}} \quad \text{(as H-modules)}. \quad (4.3)$$

It follows that there are centralizer algebras isomorphisms:

$$Z_k(G) := \text{End}_G(V^{\otimes k}) \cong \text{End}_G(U^k),$$

$$Z_{k+\frac{1}{2}}(H) := \text{End}_H(\text{Res}_H^G(V^{\otimes k})) \cong \text{End}_H(U^{k+\frac{1}{2}}). \quad (4.4)$$

Suppose for $k \in \mathbb{Z}_{\geq 0}$ that

- $\Lambda_{k,G} \subseteq \Lambda_G$ indexes the irreducible $G$-modules, and hence also the irreducible $Z_k(G)$-modules, in $U^k \cong V^{\otimes k}$;

- $\Lambda_{k+\frac{1}{2},H} \subseteq \Lambda_H$ indexes the irreducible $H$-modules, and hence also the irreducible $Z_{k+\frac{1}{2}}(H)$-modules, in $U^{k+\frac{1}{2}} \cong \text{Res}_H^G(V^{\otimes k})$.

The restriction-induction Bratteli diagram for the pair $(G, H)$ is an infinite, rooted tree $\mathcal{B}(G, H)$ whose vertices are organized into rows labeled by half integers $\ell$ in $\frac{1}{2}\mathbb{Z}_{\geq 0}$. For $\ell = k \in \mathbb{Z}_{\geq 0}$, the vertices on row $k$ are the elements of $\Lambda_{k,G}$, and the vertices on row $\ell = k + \frac{1}{2}$ are the elements of $\Lambda_{k+\frac{1}{2},H}$. The vertex on row 0 is the root, the label of the trivial $G$-module, and the vertex on row $\frac{1}{2}$ is the label of the trivial $H$-module. For the pair $(S_n, S_{n-1})$ (or $(A_n, A_{n-1})$), the labels on rows 0 and $\frac{1}{2}$ are the partitions $[n], \ [n-1]$ having just one part.

The edges of $\mathcal{B}(G, H)$ are given by drawing $c_\alpha^\lambda$ edges from $\lambda \in \Lambda_{k,G}$ to $\alpha \in \Lambda_{k+\frac{1}{2},H}$, where $c_\alpha^\lambda$ is as in (4.1). Similarly, there are $c_\beta^\kappa$ edges from $\beta \in \Lambda_{k+\frac{1}{2},H}$ to $\kappa \in \Lambda_{k+1,G}$. The Bratteli diagram is constructed in such a way that

- the number of paths from the root at level 0 to $\lambda \in \Lambda_{k,G}$ equals the multiplicity of $G^\lambda$ in $U^k \cong V^{\otimes k}$ and thus also equals the dimension of the irreducible $Z_k(G)$-module $Z_{U^k,G}^\lambda$ (these numbers are computed in Pascal-triangle-like fashion and are placed below each vertex);
• the number of paths from the root at level 0 to \( \alpha \in \Lambda_{k+\frac{1}{2},H} \) equals the multiplicity of \( H^\alpha \) in \( U^{k+\frac{1}{2}} \) and thus also equals the dimension of the \( Z_{k+\frac{1}{2}}(H) \)-module \( Z^{\alpha}_{U^{k+\frac{1}{2}},H} \) (and is indicated beneath each vertex);

• the sum of the squares of the labels on row \( k \) (resp. row \( k + \frac{1}{2} \)) equals \( \dim Z_k(G) \) (resp. \( \dim Z_{k+\frac{1}{2}}(H) \)).

When \((G,H) = (S_n, S_n)\) or when \((G,H) = (A_n, A_{n-1})\), it is well known (and easy to verify) that the permutation module satisfies \( M_n \cong U^1 = \text{Ind}_{S_{n-1}} S_n (S_n^{[n]}) \). Then (4.4) implies there are partition algebra surjections as \( P_k(n) \to Z_k(n) = \text{End}_{S_n} (M_n \otimes^k) \cong \text{End}_{S_n} (U^k) \) and \( P_{k+\frac{1}{2}}(n) \to Z_{k+\frac{1}{2}}(n) = \text{End}_{S_{n-1}} (M_n \otimes^k) \cong \text{End}_{S_{n-1}} (U^k) \). Using the restriction/induction rules for \( S_n \) in (3.3) and for \( A_n \) in (3.4), we construct the Bratteli diagram for \((S_4, S_3)\) (see Figure 1) and for \((A_4, A_3)\) (see Figure 2). In Appendices A.1 and A.3 we construct the Bratteli diagrams for \((S_6, S_5)\) and \((A_6, A_5)\).

![Bratteli Diagram](image)

Figure 1: Levels \( \ell = 0, \frac{1}{2}, 1, \ldots, \frac{7}{2}, 4 \) of the Bratteli diagram for the pair \((S_4, S_3)\).

**Remark 4.5.** Amazingly, the Bratteli diagrams in Figures 1 and 2 also appear in the Schur-Weyl duality analysis of the McKay correspondence, as discussed in [B] and [BH2]. The binary octahedral subgroup \( O \) of the special unitary group \( SU_2 \) is the two-fold cover of the octahedral group.
which is isomorphic to the symmetric group $S_4$. We use that fact to show that the Bratteli diagram for tensor powers of the 2-dimensional spin module of $O$ (which is the defining module of $O$ and $SU_2$) is identical to Figure 1. Similarly, the binary tetrahedral subgroup $T$ is the two-fold cover of tetrahedral group, which is isomorphic to the alternating group $A_4$. The Bratteli diagram for tensor powers of the 2-dimensional defining module of $T$ is identical to Figure 2.

Remark 4.6. The tensor power Bratteli diagram $B_V(G)$ is constructed using the centralizer algebras $Z_k(G) = \text{End}_G(V^\otimes k)$. The vertices on level $k$ of $B_V(G)$ are labeled by elements of $\Lambda_k,G$, and there are $c_{\mu}^\lambda$ edges from $\lambda \in \Lambda_k,G$ to $\mu \in \Lambda_{k+1},G$ if $G^\lambda \otimes V \cong \bigoplus_{\mu \in \Lambda_k} c_{\mu}^\lambda G^\mu$. In the special case that $V = \text{Ind}_H^G(\text{Res}_H^G(U^0))$ and $U^0$ is the trivial $G$-module, $B_V(G)$ is identical to $B(G,H)$ except that the half integer levels are missing from $B_V(G)$. So for example, in the tensor power Bratteli diagram that corresponds to Figure 1 there are two edges from the vertex $\square$ on level 1 to the vertex $\square$ on level 2. Including the intermediate half-integer levels, which corresponds to performing restriction and then induction, results in a diagram without multiple edges between vertices when $(G,H) = (S_n,S_{n-1})$ or $(A_n,A_{n-1})$, since the restriction/induction rules for those pairs are multiplicity free. The half-integer centralizer algebras have proven to be a powerful tool in studying the structure of these tensor power centralizer algebras (for example, in [HR] and [BH2]), and we use them here to recursively derive dimension formulas.
5 Dimensions formulas for symmetric group centralizer algebras

In the next two sections, we determine expressions for the dimensions of the irreducible modules for the centralizer algebras in Table 1. Our arguments will invoke standard combinatorial facts about representations of the symmetric group \( S_n \). The dimensions will be expressed as integer combinations of Stirling numbers of the second kind. We begin by briefly reviewing some known results about these numbers.

5.1 Stirling numbers of the second kind and Bell numbers

There are several commonly used notations for Stirling numbers of the second kind; for example, \( S(\binom{k}{t}) \) is used by Stanley [S]. In [K], Knuth remarks “The lack of a widely accepted way to refer to these numbers has become almost scandalous,” and he goes on to make a convincing argument for adopting the notation \( \{ \binom{k}{t} \} \), which we will do here.

The Stirling number \( \{ \binom{k}{t} \} \) of the second kind counts the number of ways to partition a set of \( k \) elements into \( t \) disjoint nonempty blocks. In particular, \( \{ \binom{k}{0} \} = 0 \) for all \( k \geq 1 \), and \( \{ \binom{k}{t} \} = 0 \) if \( t > k \). By convention, \( \{ \binom{0}{0} \} = 1 \). These numbers satisfy the recurrence relations,

\[
\begin{align*}
\{ \binom{k+1}{t+1} \} &= \sum_{r=t}^{k} \binom{\binom{k}{r}}{\binom{r}{t}} \\
\{ \binom{k+1}{t} \} &= t \{ \binom{k}{t} \} + \{ \binom{k}{t-1} \}.
\end{align*}
\]  

(5.1)

(5.2)

For \( k \geq 1 \),

\[
\sum_{t=0}^{k} \{ \binom{k}{t} \} = \sum_{t=1}^{k} \{ \binom{k}{t} \} = B(k),
\]

(5.3)

where \( B(k) \) is the \( k \)th Bell number. More generally, for \( k \geq 1 \),

\[
\sum_{t=1}^{n} \{ \binom{k}{t} \} = B(k, n)
\]

(5.4)

counts the number of ways to partition a set of \( k \) elements into at most \( n \) disjoint nonempty blocks, and \( B(k, n) = B(k) \) if \( n \geq k \). Identifying \( P_0(\xi) \) with \( \mathbb{C} \), we have \( \dim P_k(\xi) = B(2k) \) for all \( k \in \mathbb{Z}_{\geq 0} \). In fact, \( \dim P_\ell(\xi) = B(2\ell) \) for all \( \ell \in \frac{1}{2} \mathbb{Z}_{\geq 0} \), which can be seen by taking \( \nu = \emptyset \) in Corollary 5.26 below.

5.2 \( r \)-sequences for \( \lambda \)

In what follows, \( \lambda = [\lambda_1, \ldots, \lambda_n] \) is a partition of \( n \), \( \ell(\lambda) \) is the number of nonzero parts of \( \lambda \), and \( f^\lambda \) is the dimension of the irreducible \( S_n \)-module \( S^\lambda_n \), that is, the number of standard Young tableaux of shape \( \lambda \). We define \( f^\emptyset \) to be 1 for the unique partition \( \emptyset \) of 0. Since the rule for restricting \( S^\lambda_n \) to \( S_{n-1} \) is given by \( S^\lambda_n = \bigoplus_{\mu=\lambda-\Box} f^\mu S^\mu_{n-1} \), it follows that \( f^\lambda = \sum_{\mu=\lambda-\Box} f^\mu \). When \( \mu \subseteq \lambda \), we let \( f^{\lambda/\mu} \) denote the number of standard tableaux of skew shape \( \lambda/\mu \). We set \( \lambda^\# = [\lambda_2, \ldots, \lambda_n] \), the partition obtained from \( \lambda \) by removing the first (largest) part, so that \( |\lambda^\#| = n - \lambda_1 \).
Definition 5.5. For $r = 1, \ldots, \ell(\lambda)$, say a sequence $\underline{\pi} = \{\pi^{(0)}, \pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(r)}\}$ of $r + 1$ partitions is an $r$-sequence for $\lambda$ if

(1) $\pi^{(0)} = [\lambda_r, 0, \ldots, 0] \subseteq \pi^{(1)} \subseteq \pi^{(2)} \subseteq \ldots \subseteq \pi^{(r)} = \lambda$, and

(2) $\pi_1^{(1)} = \pi_2^{(1)} = \pi_3^{(2)} = \ldots = \pi_{r-1}^{(r-1)} = \pi_r^{(r-1)} = \lambda_r$.

In stating this definition, we are assuming for each partition $\pi^{(i)}$ in the $r$-sequence $\underline{\pi}$ for $\lambda$ that $\pi^{(i)} = [\pi_1^{(i)}, \ldots, \pi_n^{(i)}]$, so that the parts of $\pi^{(i)}$ are $\pi_j^{(i)}$, $j = 1, \ldots, n$.

Example 5.6. If $\lambda = [5, 5, 4, 2]$, then $\underline{\pi} = \{[4, 0, 0, 0], [4, 4, a, b], [c, 4, 4, d], [5, 5, 4, 2]\}$ is a 3-sequence for $\lambda$ for all $0 \leq a \leq 4$, $0 \leq b \leq \min(a, 2)$, $b \leq d \leq 2$, and $4 \leq c \leq 5$.

Definition 5.7. Let $\lambda = [\lambda_1, \ldots, \lambda_{\ell(\lambda)}]$ be a partition of $n$ with $\ell(\lambda)$ nonzero parts. Set $F^\lambda_0 := 0$, and for $1 \leq r \leq \ell(\lambda)$ define

$$F^\lambda_r := \begin{cases} 0 & \text{if } \lambda_r \leq r, \\ \sum_{\underline{\pi} \text{ is an } r\text{-sequence for } \lambda} f^\lambda_{\underline{\pi}} & \text{if } \lambda_r > r \geq 1, \end{cases} \quad (5.8)$$

where if $\underline{\pi} = \{\pi^{(0)}, \pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(r)}\}$ is an $r$-sequence for $\lambda$, then $f^\lambda_{\underline{\pi}} := \prod_{j=1}^{r} f^{\pi^{(j)}/\pi^{(j-1)}}$.

It is a consequence of this definition that the following recursive relation holds (which is an equivalent way to define $F^\lambda_r$),

$$F^\lambda_r = \begin{cases} 0 & \text{if } \lambda_r \leq r, \\ f^{\lambda/\mu}_{\mu} \sum_{\mu \subseteq \lambda, \mu_{r-1} = \mu_r = \lambda_r} F^{\mu-1}_{r-1} \cdot f^{\lambda/\mu} & \text{if } r = 1 \text{ and } \lambda_1 > 1, \\ \sum_{\mu \subseteq \lambda, \mu_{r-1} = \lambda_r} F^{\mu-1}_{r-1} \cdot f^{\lambda/\mu} & \text{if } \lambda_r > r > 1. \end{cases} \quad (5.9)$$

Remark 5.10. If $\underline{\pi}$ is an $r$-sequence for $\lambda$ then, by definition, $f^\lambda_{\underline{\pi}}$ equals the number of standard Young tableaux of shape $\lambda$ with the property that $\pi^{(j)}$ contains the numbers $1, \ldots, |\pi^{(j)}|$ for each $0 \leq j \leq r$. We say that such a tableau is compatible with $\underline{\pi}$ or is $\underline{\pi}$-compatible. Thus for $r \geq \lambda_r$, $F^\lambda_r$ equals the number of standard tableaux of shape $\lambda$ that are compatible with $\underline{\pi}$, as $\underline{\pi}$ varies over all possible $r$-sequences for $\lambda$. Example 5.11 examines $F^\lambda_r$ for $\lambda = [5, 4, 1]$ and $r = 2$. Further examples can be found in Appendix B, and tables of values of $F^\lambda_r$ for all partitions $\lambda$ of $n \leq 12$ are given in Appendix C.

Example 5.11. If $\lambda = [5, 4, 1]$ and $r = 2$, then $F^\lambda_2 = 6$. There are two 2-sequences for $\lambda$, namely $\underline{\pi}_1 = \{[4, 0, 0], [4, 4, 0], [5, 4, 1]\}$ and $\underline{\pi}_2 = \{[4, 0, 0], [4, 4, 1], [5, 4, 1]\}$. There are two standard tableaux compatible with $\underline{\pi}_1$,

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & & &
\end{array}
\quad \text{and} \quad
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & & &
\end{array}
\]
and four compatible with \( \varnothing \).

Observe that the same standard tableau can get counted more than once in \( F_r^\lambda \) if it is compatible with two different \( r \)-sequences, as happens in this example.

In the statement of next proposition, \( \mu = \lambda - \square_i \) signifies a box has been removed from row \( i \) to obtain the partition \( \mu \). If removing a box from row \( i \) fails to give a partition, the corresponding term \( F_r^\mu \) is assumed to equal 0.

**Proposition 5.12.** Suppose \( \lambda \) is a partition of \( n \) and \( 1 \leq r \leq \ell(\lambda) \). Then

\[
F_r^\lambda = \delta_{\lambda_{r-1}, \lambda_r} F_{r-1}^\lambda + \sum_{\mu=\lambda-\square_i, i \neq r} F_r^\mu \quad (\text{where } \lambda_0 := 0).
\]

(b) If \( \lambda_r > \lambda_{r+1} \) and \( \lambda_r > r + 1 \), then \( F_r^\lambda = F_r^{\lambda-\square_r} \).

**Proof.** (a) We know that \( F_r^\lambda \) enumerates all standard tableaux of shape \( \lambda \) which are compatible with some \( r \)-sequence \( \pi \) for \( \lambda \). We show that the right-hand side of this proposition counts these same tableaux but organized according to the location of the largest entry \( n \) in the tableau.

Let \( T \) be a standard tableau of shape \( \lambda \) that is compatible with the \( r \)-sequence \( \pi \) for \( \lambda \). The largest entries of \( T \) are in the skew shape \( \lambda/\pi^{(r-1)} \), so the only way for \( n \) to lie in \( \pi^{(r-1)} \) is if \( \pi^{(r-1)} = \pi^{(r)} = \lambda \). In this case, \( \lambda_{r-1} = \pi_{r-1}^{(r-1)} = \pi_r^{(r-1)} = \lambda_r \), so \( \lambda_{r-1} = \lambda_r \) must hold. The tableaux \( T \) with \( n \) in \( \pi^{(r-1)} \) are compatible with the \( (r-1) \)-sequence \( \{\pi(0), \pi(1), \ldots, \pi^{(r-1)} = \lambda\} \), and the number of such tableaux is \( F_{r-1}^\lambda \).

Alternatively, the largest entry \( n \) of \( T \) is in \( \lambda \setminus \pi^{(r-1)} \). Since \( \pi_r^{(r-1)} = \lambda_r \), this entry is not in the \( r \)th row of \( \lambda \). Furthermore, since \( n \) is the largest entry, it must be placed in a “removable” box of \( \lambda \); that is, the rightmost entry in a row \( \square_i \) with \( \lambda_i > \lambda_{i+1} \). The number of such tableaux \( T \) is counted by \( F_r^{\lambda-\square_i} \), and the proposition is proved by summing over all such \( i \).

(b) A proof of this part is given in Appendix B in order not to break up the flow of this section. \( \square \)

### 5.3 Main result for symmetric group centralizer algebras

Our aim in this section is to establish Theorem 5.13, which gives the dimensions of the irreducible modules for the centralizer algebras \( Z_k(n) = \text{End}_{S_n}(M_n^{\otimes k}) \) and \( Z_{k+\frac{1}{2}}(n) = \text{End}_{S_{n-1}}(M_n^{\otimes k}) \).

**Theorem 5.13.** Let \( k, n \in \mathbb{Z}_{\geq 0} \) and \( n \geq 1 \), and let the notation be as in Table 7

(a) Assume \( \lambda = [\lambda_1, \ldots, \lambda_n] \vdash n \), and \( \lambda \in \Lambda_{k, S_n} \). Then

\[
\dim Z_{k,n}^\lambda = \sum_{r=1}^{\ell(\lambda)} (-1)^{r-1} F_r^\lambda \left( \sum_{t=n-\lambda_r+r-1}^{n-2} \binom{n-\lambda_r+r-1}{t} \binom{k}{t} \right) + f^\lambda \left[ \binom{k}{n-1} + \binom{k}{n} \right].
\]
(b) Assume $\mu = [\mu_1, \ldots, \mu_{n-1}] \vdash n-1$, and $\mu \in \Lambda_{k+\frac{1}{2}} S_{n-1}$. Then

$$\dim Z^\mu_{k+\frac{1}{2}, n} = \sum_{r=1}^{l(\mu)} (-1)^{r-1} F^\mu_r \left( \sum_{t=n-\mu_r + r-2}^{n-3} \binom{t + 1}{\mu_r + r - 2} \right) \binom{k + 1}{t + 1} + f^\mu \left( \left\{ \frac{k + 1}{n - 1} \right\} + \left\{ \frac{k + 1}{n} \right\} \right).$$

(c) $\dim Z_k(n) = \dim Z_{2k,n}^{[n]} = \sum_{t=0}^{n} \binom{2k}{t} = B(2k, n) \quad (= B(2k) \text{ if } n \geq 2k).$

(d) $\dim Z_{k+\frac{1}{2}}(n) = \dim Z_{2k+\frac{1}{2}, n}^{[n-1]} = \sum_{t=0}^{n-3} \binom{2k + 1}{t + 1} + \left( \binom{2k + 1}{n - 1} + \binom{2k + 1}{n} \right) \right) = \sum_{t=1}^{n} \binom{2k + 1}{t} = B(2k + 1, n) \quad (= B(2k + 1) \text{ if } n \geq 2k + 1).$

**Proof.** Our proof is by induction on $k \in \mathbb{Z}_{\geq 0}$. For any $n \geq 1$, one can confirm that part (a) gives

$$\dim Z^\lambda_{0,n} = \begin{cases} 1 & \text{if } \lambda = [n] \\ 0 & \text{otherwise,} \end{cases} \quad (5.14)$$

$$\dim Z^\lambda_{1,n} = \begin{cases} 1 & \text{if } \lambda = [n] \text{ or } [n - 1, 1] \text{ when } n \geq 2, \text{ and } \lambda = [1] \text{ when } n = 1, \\ 0 & \text{otherwise,} \end{cases}$$

which are the correct values for these dimensions, so part (a) holds for $k = 0, 1$. We assume (a) holds for $k$ and for all $n \geq 1$ and argue that (b) holds for $k$. We then use that result to show (a) holds for $k + 1$.

As an $S_{n-1}$-module, $M_n = M_{n-1} \oplus \mathbb{C} u_n$ where $\sigma u_n = u_n$ for all $\sigma \in S_{n-1}$ and $M_{n-1}$ is the permutation module for $S_{n-1}$. It follows that $M_n^{\otimes k} \cong \sum_{s=0}^{k} \binom{k}{s} M_{n-1}^{\otimes s}$ as $S_{n-1}$-modules. Applying that result and the inductive hypothesis, we obtain for $\mu$ a partition of $n - 1$, 

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\[ \dim Z^\mu_{k+\n} = \sum_{s=0}^{k} \binom{k}{s} \dim Z^\mu_{s,n-1} \]
\[ = \sum_{s=0}^{k} \binom{k}{s} \left( \sum_{r=1}^{\ell(\mu)} (-1)^{r-1} F^\mu_r \left( \sum_{t=n-\mu_r+r-2}^{n-3} \binom{t}{n-\mu_r+r-2} \binom{k}{s} \left\{ \binom{s}{t} \right\} \right) \right) \]
\[ + f^\mu \sum_{s=0}^{k} \binom{k}{s} \left( \left\{ \binom{s}{n-2} \right\} + \left\{ \binom{s}{n-1} \right\} \right) \]
\[ = \sum_{r=1}^{\ell(\mu)} (-1)^{r-1} F^\mu_r \left( \sum_{t=n-\mu_r+r-2}^{n-3} \binom{t}{n-\mu_r+r-2} \binom{k}{s} \left\{ \binom{s}{t} \right\} \right) \]
\[ + f^\mu \left( \left\{ \binom{k+1}{n-1} \right\} + \left\{ \binom{k+1}{n} \right\} \right) \]
\[ = \sum_{r=1}^{\ell(\mu)} (-1)^{r-1} F^\mu_r \left( \sum_{t=n-\mu_r+r-2}^{n-3} \binom{t}{n-\mu_r+r-2} \binom{k+1}{s+1} \left\{ \binom{s+1}{t+1} \right\} \right) \]
\[ + f^\mu \left( \left\{ \binom{k+1}{n-1} \right\} + \left\{ \binom{k+1}{n} \right\} \right) \]
using (5.2),

which establishes the formula in (b) for \( k \) and all \( n \geq 1 \).

Now assume \( \lambda \vdash n \) as in (a). Restricting the representation from \( Z_{k+1}(n) \) to \( Z_{k+\frac{1}{2}}(n) \) gives \( \dim Z^\lambda_{k+1,n} = \sum_{\mu=\lambda-\square} \dim Z^\mu_{k+\frac{1}{2},n} \), where the sum is over all partitions \( \mu \) of \( n-1 \) obtained by removing a box from \( \lambda \) (see Thm. 2.7 (b)). Therefore,

\[ \dim Z^\lambda_{k+1,n} = \sum_{\mu=\lambda-\square} \dim Z^\mu_{k+\frac{1}{2},n} \]
\[ = \sum_{\mu=\lambda-\square} \sum_{r=1}^{\ell(\lambda)} (-1)^{r-1} F^\mu_r \left( \sum_{s=n-\mu_r+r-2}^{n-3} \binom{s}{n-\mu_r+r-2} \binom{k+1}{s+1} \left\{ \binom{s+1}{t+1} \right\} \right) \]
\[ + \sum_{\mu=\lambda-\square} f^\mu \left( \left\{ \binom{k+1}{n-1} \right\} + \left\{ \binom{k+1}{n} \right\} \right) \]

(Note that \( F^\mu_r = 0 \) for all \( r > \ell(\mu) \), so there is no harm in summing to \( \ell(\lambda) \geq \ell(\mu) \).)

Since \( \sum_{\mu=\lambda-\square} f^\mu \left( \left\{ \binom{k+1}{n-1} \right\} + \left\{ \binom{k+1}{n} \right\} \right) = f^\lambda \left( \left\{ \binom{k+1}{n-1} \right\} + \left\{ \binom{k+1}{n} \right\} \right) \), it suffices to show that the first line of (5.15) equals \( \sum_{r=1}^{\ell(\lambda)} (-1)^{r-1} F^\lambda_r \left( \sum_{t=n-\lambda_r+r-1}^{n-3} \binom{t}{n-\lambda_r+r-1} \left\{ \binom{t}{k+1} \right\} \right) \).

Now if \( \lambda_r = \lambda_{r+1} \), then by definition \( F^\lambda_r = 0 \). If \( \lambda_r > \lambda_{r+1} \) and \( \lambda_r > r+1 \), then by Proposition 5.12 (b), \( F^\lambda_r = \delta_{\lambda_r, \lambda_{r+1}} F^\lambda_r + F^\lambda_{r-\square} \). Thus,

\[ F^\lambda_r = \delta_{\lambda_r, \lambda_{r+1}} F^\lambda_r + F^\lambda_{r-\square} \]
holds whenever \( \lambda_r > r+1 \). We use that observation along with Proposition 5.12 (a) in computing the following summation. Let \( 1 \leq t \leq n-1 \), and for the time being assume that \( \lambda_r > r+1 \) for
all \(1 \leq r \leq \ell(\lambda)\). Then for a fixed value of \(t\), we consider the coefficient of \((k+1)_t\) in the above expression:

\[
\sum_{r=1}^{\ell(\lambda)} (-1)^{r-1} \sum_{\mu=\lambda-\square} F_{r}^{\mu} \left( n - \mu_{r} + r - 2 \right)
\]

\[
= \sum_{r=1}^{\ell(\lambda)} (-1)^{r-1} \sum_{\mu=\lambda-\square} F_{r}^{\mu} \left( n - \mu_{r} + r - 2 \right) + \sum_{r=1}^{\ell(\lambda)} (-1)^{r-1} F_{r}^{\lambda-\square_{r}} \left( n - \lambda_{r} + r - 1 \right)
\]

\[
= \sum_{r=1}^{\ell(\lambda)} (-1)^{r-1} \left( F_{r}^{\lambda} - \delta_{\lambda_{r-1},\lambda_{r}} F_{r-1}^{\lambda} \right) \left( n - \lambda_{r} + r - 2 \right) + \sum_{r=1}^{\ell(\lambda)} (-1)^{r-1} F_{r}^{\lambda-\square_{r}} \left( n - \lambda_{r} + r - 1 \right)
\]

\[
= \sum_{r=1}^{\ell(\lambda)} (-1)^{r-1} F_{r}^{\lambda} \left( n - \lambda_{r} + r - 2 \right) - \sum_{r=1}^{\ell(\lambda)} (-1)^{r-1} \delta_{\lambda_{r-1},\lambda_{r}} F_{r-1}^{\lambda} \left( n - \lambda_{r} + r - 2 \right)
\]

\[
+ \sum_{r=1}^{\ell(\lambda)} (-1)^{r-1} F_{r}^{\lambda-\square_{r}} \left( n - \lambda_{r} + r - 1 \right)
\]

\[
= \sum_{r=1}^{\ell(\lambda)} (-1)^{r-1} F_{r}^{\lambda} \left( n - \lambda_{r} + r - 2 \right) + \sum_{q=1}^{\ell(\lambda)-1} (-1)^{q-1} \delta_{\lambda_{q},\lambda_{q+1}} F_{q}^{\lambda} \left( n - \lambda_{q} + q - 1 \right)
\]

\[
+ \sum_{r=1}^{\ell(\lambda)} (-1)^{r-1} F_{r}^{\lambda-\square_{r}} \left( n - \lambda_{r} + r - 1 \right)
\]

\[
= \sum_{r=1}^{\ell(\lambda)} (-1)^{r-1} \left( F_{r}^{\lambda} \left( n - \lambda_{r} + r - 2 \right) + \delta_{\lambda_{r},\lambda_{r+1}} F_{r}^{\lambda} + F_{r}^{\lambda-\square_{r}} \right) \left( n - \lambda_{r} + r - 1 \right)
\]

\[
= \sum_{r=1}^{\ell(\lambda)} (-1)^{r-1} F_{r}^{\lambda} \left( n - \lambda_{r} + r - 2 \right) + F_{r}^{\lambda} \left( n - \lambda_{r} + r - 1 \right) \text{ by } (5.16)
\]

\[
= \sum_{r=1}^{\ell(\lambda)} (-1)^{r-1} F_{r}^{\lambda} \left( n - \lambda_{r} + r - 1 \right).
\]

The above calculation shows that the desired result holds when \(\lambda_{r} > r + 1\). Now if \(\lambda_{r} = r + 1\) then \((n-\lambda_{r}+r-1) = (n-r-1) = 0\) for all \(1 \leq t \leq n - 2\). Thus, the third from last line in the previous summation becomes \(\sum_{r=1}^{\ell(\lambda)} (-1)^{r-1} F_{r}^{\lambda} \left( n-3 \right)\). For \(t = n - 2\), this is \(\sum_{r=1}^{\ell(\lambda)} (-1)^{r-1} F_{r}^{\lambda} \left( n-3 \right) = \sum_{r=1}^{\ell(\lambda)} (-1)^{r-1} F_{r}^{\lambda} \left( n-2 \right)\), which is the desired conclusion. Finally, if \(\lambda_{r} < r+1\), then since \(t \leq n-2\), the terms \((n-\lambda_{r}+r-1) = 0\) are all zero, so the statement in (a) holds in this case as well. Hence part (a) of Theorem 5.13 is true for \(k+1\), and by induction, (a) holds for all \(k \geq 0\) and all \(n \geq 1\).

Part (c) is an immediate consequence of (a), since \(S_{n}^{[n]}\) is the trivial \(S_{n}\)-module, and \(M_{\mu}^{n}k\) is isomorphic, as an \(S_{n}\)-module, to its dual module, so that
Part (d) follows readily from (b) for similar reasons.

Remark 5.17. In [D] Prop. 2.1, it was shown using the exponential generating functions of Goupil and Chauve [GC] that the multiplicity of $S^n_\lambda$ in $M_n^{\otimes k}$ (i.e. $\dim Z_{k,n}^\lambda$) equals $F_1^\lambda \sum_{t=|\lambda^\#|}^{n-2} \binom{t}{k} \binom{n}{n-1} + \binom{n}{n}$ whenever $1 \leq k \leq n - \lambda_2$. This is a special case of part (a) of Theorem 5.13 since the only nonzero term that occurs in (a) under these assumptions is when $r = 1$. As mentioned earlier, this result was used in [D] to bound the mixing time of a Markov chain on $S_n$.

Example 5.18. If $n = 9$ and $\lambda = [5, 4, 4]$ then Theorem 5.13(a) gives

$$\dim Z_{k,n}^{[4,4,4]} = F_1^{[5,4,4]} \left( \begin{array}{c} 8 \\ 8 \end{array} \right) \left( \begin{array}{c} k \\ 8 \end{array} \right) + \left( \begin{array}{c} 9 \\ 8 \end{array} \right) \left( \begin{array}{c} k \\ 9 \end{array} \right) + \left( \begin{array}{c} 10 \\ 8 \end{array} \right) \left( \begin{array}{c} k \\ 10 \end{array} \right) + \left( \begin{array}{c} 11 \\ 8 \end{array} \right) \left( \begin{array}{c} k \\ 11 \end{array} \right)$$

which equals 14, 630, 16710, 343002, 6052134, 96860478, 1451113664, for $k = 8, 9, \ldots, 14$.

Example 5.19. If $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)}]$ and $\lambda_2 \leq 2$, then $F_2^\lambda = 0$ by Definition 5.7. In this case Theorem 5.13(a) simplifies to

$$\dim Z_{k,n}^\lambda = F_1^\lambda n^{-2} \binom{t}{t} \binom{k}{k} + f^\lambda \left( \begin{array}{c} k \\ n-1 \end{array} \right) + \binom{k}{n},$$

where $\lambda^\# = [\lambda_2, \ldots, \lambda_{\ell(\lambda)}]$ as above. In particular, this formula holds for all $\lambda$ when $n \leq 5$. For the special case $\lambda = [1^n]$, it gives $\dim Z_{k,n}^{[1^n]} = \binom{k}{n-1} + \binom{k}{n}$ for all $n \geq 1$.

Remark 5.20. Suppose $\nu = [\nu_1, \ldots, \nu_{\ell(\nu)}]$ is a partition with $0 \leq |\nu| \leq k$, and for $n \geq 2k$, let $[n - |\nu|, \nu]$ be the partition of $n$ given by $[n - |\nu|, \nu] := [n - |\nu|, \nu_1, \ldots, \nu_{\ell(\nu)}]$. In the next proposition, we obtain an expression for the dimension of the irreducible $Z_k(n)$-module $Z_{k,n}^{[n - |\nu|, \nu]}$ (and for the $Z_{k+\frac{k}{2}}(n)$-module $Z_{k+\frac{k}{2},n}^{[n - |\nu|, \nu]}$ when $n - 1 \geq 2k$). We prove that both dimensions equal
\[ f' = \dim S_k^{\nu} \text{ when } |\nu| = k. \] When \( \nu = [k] \), \( \dim Z_{k,n}^{[n-k,k]} = f[k] = 1 \) for all \( n \geq k \). When \( n = 2k - 1 \), the kernel of the map \( P_k(n) \to Z_k(n) \) is one-dimensional, since \( [n - k, k] \) is not a partition in that case. In \([BH1]\), we describe the kernel of the map \( P_k(n) \to Z_k(n) \) for all \( n < 2k \).

**Proposition 5.21.** Assume \( \nu = [\nu_1, \ldots, \nu_d(\nu)] \) is a partition with \( 0 \leq |\nu| \leq k \).

(a) If \( 0 \leq 2k \leq n \), then
\[
\dim Z_{k,n}^{[n-|\nu|,\nu]} = f' \sum_{t=|\nu|}^{k} \binom{t}{\nu} \binom{k}{t} \quad (= f' \text{ when } |\nu| = k). \quad (5.22)
\]

(b) If \( 0 \leq 2k \leq n - 1 \), then
\[
\dim Z_{k+\frac{1}{2},n}^{[n-1-|\nu|,\nu]} = f' \sum_{t=|\nu|}^{k} \binom{t}{\nu} \binom{k+1}{t+1} \quad (= f' \text{ when } |\nu| = k). \quad (5.23)
\]

**Proof.** (a) Because \( k + r - 1 \leq n - k + r - 1 \leq n - |\nu| + r - 1 \leq n - \nu + r - 1 \), for the partition \( \lambda := [n - |\nu|, \nu] \) all the terms in Theorem 5.13(a) with \( r \geq 2 \) vanish, and
\[
\dim Z_{k,n}^{\lambda} = F^{\lambda}_{1} \sum_{t=|\nu|}^{n-2} \binom{t}{\nu} \binom{k}{t} + f^{\lambda} \left( \binom{k}{n-1} + \binom{k}{n} \right). \quad (5.24)
\]

Consider first the case that \( k = 0 \). Then \( \nu = \emptyset \), and (5.24) reduces to \( f^{0} \) for \( n \geq 2 \), which is what (5.22) is asserting. When \( n = 1 \), then \( F^{1}_{1} = 0 \), and \( \dim Z_{0,1}^{[0]} = f^{[1]} \{ \emptyset \} = f^{[1]} = 1 \), as in (5.22). Consider next the possibility \( k = 1 \). Then \( \nu = \emptyset \) or \([1]\), and the following shows that (5.22) holds:
\[
\dim Z_{1,n}^{[n-|\nu|,\nu]} = \begin{cases} f' \sum_{t=|\nu|}^{k} \binom{t}{\nu} \binom{k}{t} & \text{if } n \geq 3, \\ f' \{1\} & \text{if } n \geq 1,2. \end{cases}
\]

Finally, for \( k \geq 2 \), we have \( n \geq 2k > k + 1 \geq |\nu| + 1 \), so that \( n - |\nu| > 1 \) and \( F^{\lambda}_{1} = f^{\nu} \). There is no contribution from the right-hand summand in (5.24), and (5.22) is valid in this case as well.

(b) Replacing \( n \) with \( n - 1 \), we see as in (a) that \( k + r - 1 \leq n - 1 - \nu + r - 1 = n - \nu + r - 2 \). This implies that all the terms in Theorem 5.13(b) for \( \mu = [n - 1 - |\nu|, \nu] \) are 0 for \( r \geq 2 \), and therefore
\[
\dim Z_{k+\frac{1}{2},n}^{\mu} = F^{\mu}_{1} \sum_{t=k}^{n-3} \binom{t}{k} \binom{k+1}{t+1} + f^{\mu} \left( \binom{k+1}{n-1} + \binom{k+1}{n} \right). \quad (5.25)
\]

Under the assumption \( 0 \leq 2k \leq n - 1 \), the term \( \binom{k+1}{n-1} \) is nonzero only when \( k = 0 \) and \( n = 1 \). In that case, all the other terms in (5.25) are 0, and \( \dim Z_{k+\frac{1}{2},n}^{\mu} = f^{[1]} \{1\} \), which is what (5.23) says it should be. Similarly, the term \( \binom{k+1}{n-1} \) is nonzero, exactly when \( k = 1 \) and \( n = 3 \). Again all the other terms in (5.25) are 0, and \( \dim Z_{k+\frac{1}{2},n}^{\mu} = f^{[2-|\nu|,\nu]} \{1\} = f^{\nu} \{1\} \) for \( \nu = \emptyset, [1] \), which is what is claimed in (5.23). In all other instances, \( k \geq 2 \), so \( n - 1 - |\nu| \geq n - 1 - k \geq k \geq 2 \), \( F^{\mu}_{1} = f^{\nu} \), and \( \dim Z_{k+\frac{1}{2},n}^{[n-1-k,\nu]} = f^{\nu} \sum_{t=|\nu|}^{n-3} \binom{t}{t+1} \binom{k+1}{t} \) as desired. \( \square \)
The partition algebras $P_k(\xi)$ and $P_{k+\frac{1}{2}}(\xi)$ are generically semisimple for all $\xi \in \mathbb{C}$ with $\xi \not\in \{0, 1, \ldots, 2k - 1\}$ (see [MS] or [HR, Thm. 3.7]). Assume $\nu = [\nu_1, \ldots, \nu_{|\nu|}]$ is a partition with $0 \leq |\nu| \leq k$, and let $P_{k,\xi}^\nu$ denote the irreducible $P_k(\xi)$-module and $P_{k+\frac{1}{2},\xi}^\nu$ denote the irreducible $P_{k+\frac{1}{2}}(\xi)$-module indexed by $\nu$. The dimension of $P_{k,\xi}^\nu$ (resp. $P_{k+\frac{1}{2},\xi}^\nu$) is the same for all generic values of $\xi$. Therefore, we can apply Proposition [5.21] with $n = 2k$ and $\lambda = [n-|\nu|, \nu_1, \ldots, \nu_{|\nu|}] \vdash n$ to conclude the following:

**Corollary 5.26.** Let $\nu$ be a partition with $0 \leq |\nu| \leq k$. For $\xi \not\in \{0, 1, \ldots, 2k - 1\}$, let $P_{k,\xi}^\nu$ denote the irreducible $P_k(\xi)$-module and $P_{k+\frac{1}{2},\xi}^\nu$ denote the irreducible $P_{k+\frac{1}{2}}(\xi)$-module indexed by $\nu$. Then

$$
\dim P_{k,\xi}^\nu = f^\nu \sum_{t=|\nu|}^{k} \binom{t}{|\nu|} \binom{k}{t} \quad (= f^\nu \text{ when } |\nu| = k)
$$

$$
\dim P_{k+\frac{1}{2},\xi}^\nu = f^\nu \sum_{t=|\nu|}^{k} \binom{t}{|\nu|} \binom{k+1}{t+1} \quad (= f^\nu \text{ when } |\nu| = k).
$$

**Remark 5.27.** The factor $f^\nu$ that appears in Corollary 5.26 equals the number of standard Young tableaux of shape $\nu$. If $P$ is a set partition of $\{1, \ldots, k\}$ into $t$ nonempty disjoint blocks, then there are $\binom{k}{t}$ ways to choose $|\nu|$ of these blocks to be distinguished, and $B(k, |\nu|) = \sum_{t=|\nu|}^{k} \binom{t}{|\nu|} \binom{k}{t}$ counts the number of set partitions of $\{1, \ldots, k\}$ with $|\nu|$ distinguished blocks. Given a set partition $P$ with $|\nu|$ distinguished blocks, we create a standard Young tableau $T$ whose entries consist of the maximal elements of the distinguished blocks of $P$. There are $f^\nu$ ways to create $T$ with this specific set of entries.

It follows from Corollary 5.26 that $\dim P_{k,\xi}^\nu = f^\nu \cdot B(k, |\nu|)$ equals the number of pairs $(P, T)$, where $P$ is a set partition of $\{1, \ldots, k\}$ with $|\nu|$ distinguished blocks, and $T$ is a standard Young tableau of shape $\nu$ filled with the maximum entries from the distinguished blocks of $P$. In [CDDSY] Thm. 2.4], a bijection is given between such pairs $(P, T)$ and paths in the Bratteli diagram for the partition algebra $P_k(\xi)$ from the root of the diagram to the partition $\nu$ on level $k$. By Section 4, the number of these paths equals $\dim P_{k,\xi}^\nu$. The bijection of [CDDSY] works only for $P_k(n)$ for $n \geq 2k$ (that is, for generic values of $\xi$) and does not extend for $n < 2k$.

Here is a small example: Suppose $\nu = [2, 1] \vdash 3$ and $P = \{1, 2, 5 \mid 3 \mid 4, 8 \mid 6, 7, 10 \mid 9\}$ is a set partition of $k = 10$ having 3 distinguished blocks, which are the ones underlined. The maximum values in the distinguished blocks are 5, 3, 10, and there are two standard tableaux $T$ of shape $\nu$ with those entries, $\begin{array}{c} 3 \rule{10em}{0em} 5 \end{array}$ and $\begin{array}{c} 3 \rule{10em}{0em} 10 \end{array}$. The dimension of $P_{10,\xi}^\nu$ counts all possible such pairs $(P, T)$.

**5.4 Examples**

**Example 5.28.** Since $f^{[n]} = 1$, Corollary 5.26 implies for all $k \geq n$ and all generic values of $\xi$ that

$$
\dim P_{k,\xi}^{[n]} = \sum_{t=n}^{k} \binom{t}{n} \binom{k}{t} \quad \text{and} \quad \dim P_{k+\frac{1}{2},\xi}^{[n]} = \sum_{t=n}^{k} \binom{t}{n} \binom{k+1}{t+1}.
$$

(5.29)
Part (a) of Theorem 5.13 gives

\[ \dim \mathcal{Z}_{k,n}^{[n]} = \sum_{t=0}^{n} \binom{k}{t} = B(k, n) \quad (= B(k) \text{ when } n \geq k) , \quad (5.30) \]

while part (b) says

\[ \dim \mathcal{Z}_{k+\frac{1}{2},n}^{[n-1]} = \sum_{t=1}^{n} \binom{k+1}{j} = B(k+1, n) \quad (= B(k+1) \text{ when } n \geq k+1) . \quad (5.31) \]

**Remark 5.32.** A (Gelfand) model for an algebra is a module in which each irreducible module appears as a direct summand with multiplicity one. In [HRe], Halverson and Reeks construct models for certain diagram algebras, including the partition algebras \( P_k(\xi) \) for generic \( \xi \), using basis diagrams invariant under reflection about the horizontal axis (the symmetric diagrams) and the diagram conjugation action of \( P_k(\xi) \) on them. The model \( \mathcal{M}_{P_k} \) for \( P_k(\xi) \) decomposes into submodules \( \mathcal{M}_{P_k} = \bigoplus_{r,p} \mathcal{M}_{r,p}^{\nu} \), where \( \mathcal{M}_{P_k}^{\nu} = \bigoplus_{\nu \vdash r\ |\ p} \mathcal{P}_{\nu|\xi} \) according to the size \( r = |\nu| \) of the partition \( \nu \) and its number \( p = \text{odd}(\nu) \) of odd parts. By enumerating symmetric diagrams, they determine that

\[ \dim \mathcal{M}_{r,p}^{\nu} = \sum_{t=0}^{r} \binom{k}{t} (r - p - 1)!! \binom{t}{r} \binom{k}{t} \quad (5.33) \]

where \( r - p \) is even and \( (r - p - 1)!! = (r - p - 1)(r - p - 3) \cdots 3 \cdot 1 \). The factor \( \binom{r}{p} \) \( (r - p - 1)!! \) comes from the fact (see [HRe]) that

\[ \sum_{\nu \vdash r\ |\ p} f^\nu = |\Gamma^{r,p}| = \binom{r}{p} (r - p - 1)!! , \quad (5.34) \]

where \( \Gamma^{r,p} \) is the set of involutions (elements of order 2) with \( p \) fixed points in the symmetric group \( S_r \). Corollary 5.26 and (5.34) give an alternate proof of (5.33):

\[ \dim \mathcal{M}_{P_k}^{\nu} = \sum_{\nu \vdash r\ |\ p} \mathcal{P}_{\nu|\xi} = \sum_{\nu \vdash r\ |\ p} f^\nu \sum_{t=|\nu|}^{k} \binom{t}{r} \binom{k}{t} \binom{r}{p} (r - p - 1)!! \sum_{t=r}^{k} \binom{t}{r} \binom{k}{t} . \]

### 6 Dimension formulas for alternating group centralizer algebras

The restriction rules in (3.6) combined with Theorem 2.7(b) can be used to derive expressions for the dimensions of the irreducible modules for the alternating group centralizer algebras \( \hat{\mathcal{Z}}_k(n) \) and \( \hat{\mathcal{Z}}_{k+\frac{1}{2}}(n) \) from the dimension formulas for irreducible modules for \( \mathcal{Z}_k(n) \) and \( \mathcal{Z}_{k+\frac{1}{2}}(n) \) in Theorem 5.13.

**Theorem 6.1.** Assume \( k \in \mathbb{Z}_{\geq 0} \). The dimensions of the irreducible modules for \( \hat{\mathcal{Z}}_k(n) \) and \( \hat{\mathcal{Z}}_{k+\frac{1}{2}}(n) \) are as follows (using notation from Table 7).
(a) For \( \lambda \vdash n \) and \( \lambda \in \Lambda_{k,A_n}, \)
\[
\dim \widehat{Z}_k^{\lambda} = \dim Z_k^{\lambda} + \dim Z_k^{\lambda*}, \quad \text{if } \lambda > \lambda^*, \\
\dim \widehat{Z}_k^{\lambda^*} = \dim Z_k^{\lambda^*}, \quad \text{if } \lambda = \lambda^*, \\
\]
where \( \dim Z_k^{\lambda} \) and \( \dim Z_k^{\lambda^*} \) are given by the formula in Theorem 5.13(a).

(b) For \( \mu \vdash n - 1 \) and \( \mu \in \Lambda_{k,A_{n-1}}, \)
\[
\dim \widehat{Z}_{k+\frac{1}{2},n} = \dim Z_{k+\frac{1}{2},n} + \dim Z_{k+\frac{1}{2},n}^{\mu*}, \quad \text{if } \mu > \mu^*, \\
\dim \widehat{Z}_{k+\frac{1}{2},n}^{\mu^*} = \dim Z_{k+\frac{1}{2},n}^{\mu^*}, \quad \text{if } \mu = \mu^*, \\
\]
where \( \dim Z_{k+\frac{1}{2},n} \) and \( \dim Z_{k+\frac{1}{2},n}^{\mu^*} \) are given by the formula in Theorem 5.13(b).

The next corollary gives some particular instances of Theorem 6.1 of special interest.

**Corollary 6.2.** Assume \( k \in \mathbb{Z}_{>0} \) and \( r \geq 2. \) Recall the definitions of the Bell numbers \( B(k,n) \) and \( B(k) \) from (5.3) and (5.4).

(a) \( \dim \widehat{Z}_k^{[n]} = \dim Z_k^{[n]} + \dim Z_k^{[1,n]} = \sum_{t=0}^{n} \binom{k}{t} + \binom{k}{n-1} + \binom{k}{n} \)
\[ = B(k, n) + \binom{k}{n-1} + \binom{k}{n}. \]

(b) \( \dim \widehat{Z}_k(n) = \dim Z_{2k,n}^{[n]} = B(2k, n) + \binom{2k}{n-1} + \binom{2k}{n} \).

In particular, \( \dim \widehat{Z}_k(n) = B(2k) + 1 \) if \( n = 2k + 1 \) and \( \dim \widehat{Z}_k(n) = B(2k) \) if \( n > 2k + 1. \)

(c) \( \dim \widehat{Z}_{k+\frac{1}{2},n}^{[n-1]} = \dim Z_{k+\frac{1}{2},n}^{[n-1]} + \dim Z_{k+\frac{1}{2},n}^{[1,n-1]} = \sum_{j=1}^{n} \binom{k+1}{j} + \binom{k+1}{n-1} + \binom{k+1}{n} \)
\[ = B(k+1, n) + \binom{k+1}{n-1} + \binom{k+1}{n} = \dim \widehat{Z}_k^{[n]}_{k+1,n}. \]

(d) \( \dim \widehat{Z}_{k+\frac{1}{2}}(n) = \dim Z_{2k+1,n}^{[n-1]} = B(2k+1, n) + \binom{2k+1}{n-1} + \binom{2k+1}{n} \).

In particular, \( \dim \widehat{Z}_{k+\frac{1}{2}}(n) = B(2k+1) + 1 \) if \( n = 2k + 2 \), and \( \dim \widehat{Z}_{k+\frac{1}{2}}(n) = B(2k+1) \) if \( n > k + 2. \)

**Remark 6.3.** Part (b) of Corollary 6.2 was shown by Bloss [Bl1, Bl2] by different methods. Part (d) extends that result to the centralizer algebras \( \widehat{Z}_{k+\frac{1}{2}}(n) \) and gives some indication of how the algebras \( \widehat{Z}_{k+\frac{1}{2}}(n) \) “fill the gap” between the integer levels.

**Example 6.4.** Corollary 6.2(c) says that for \( k = 3 \) and \( n = 4, \)
\[
\dim \widehat{Z}_{3+\frac{1}{2}}^{[8]} = \sum_{j=1}^{4} \binom{4}{j} + \binom{4}{3} + \binom{4}{4} = 1 + 7 + 2(6 + 1) = 22.
\]
This is the subscript on the partition \([3]\) in the last row of the Bratteli diagram in Figure 2.
7 The centralizer algebra $QZ_k(n) := \text{End}_{S_n}(R_n^{\otimes k})$ for $R_n = S_n^{[n-1,1]}$
and its relatives

In [DO], Daugherty and Orellana investigated the centralizer algebra $QZ_k(n) := \text{End}_{S_n}(R_n^{\otimes k})$, where $R_n = S_n^{[n-1,1]}$, and proved that there is a variant of the partition algebra, that they termed the quasi-partition algebra and denoted $QP_k(n)$. They exhibited an algebra homomorphism $QP_k(n) \to \text{End}_{S_n}(R_n^{\otimes k})$ and showed that this mapping is always a surjection and is an isomorphism when $n \geq 2k$. The irreducible modules $QZ_{k,n}^\lambda$ for $QZ_k(n)$ are indexed by partitions $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n] \vdash n$.

In this last section, we determine a formula for the dimensions of these irreducible modules. The dimension expression we obtain holds for arbitrary values of $k$ and $n$ and differs from that in [DO, Thm. 4.6], which is valid for $n > k + \lambda_2$, and is more closely related to the one in [D, Cor. 2.2], which holds for $n \geq k + \lambda_2$. We also extend these results to the case of the corresponding centralizer algebra of the alternating group: $QZ_k(n) := \text{End}_{A_n}(R_n^{\otimes k})$.

We adopt the notation in Table 2 for various centralizer algebras and their irreducible modules associated with $R_n^{\otimes k}$. In this table, for all $k \in \mathbb{Z}_{\geq 0}$, $q\Lambda_{k,S_n}$ (resp. $q\Lambda_{k,A_n}$) is the set of indices for the irreducible $QZ_k(n)$-summands (resp. $QZ_k(n)$-summands) in $R_n^{\otimes k}$ with multiplicity at least one; similarly $q\Lambda_{k,S_{n-1}}$ (resp. $q\Lambda_{k,A_{n-1}}$) is the set of indices for the irreducible $QZ_{k+\frac{1}{2}}(n)$-summands (resp. $QZ_{k+\frac{1}{2}}(n)$-summands) in $R_n^{\otimes k}$ with multiplicity at least one.

<table>
<thead>
<tr>
<th>centralizer algebra</th>
<th>irreducible modules</th>
</tr>
</thead>
<tbody>
<tr>
<td>$QZ_k(n) := \text{End}_{S_n}(R_n^{\otimes k})$</td>
<td>$QZ_{k,n}^\lambda$, $\lambda \vdash n$, $\lambda \in q\Lambda_{k,S_n} \subseteq \Lambda_{S_n}$</td>
</tr>
<tr>
<td>$QZ_{k+\frac{1}{2}}(n) := \text{End}<em>{S</em>{n-1}}(R_n^{\otimes k})$</td>
<td>$QZ_{k+\frac{1}{2},n}^\mu$, $\mu \vdash n - 1$, $\mu \in q\Lambda_{k+\frac{1}{2},S_{n-1}} \subseteq \Lambda_{S_{n-1}}$</td>
</tr>
<tr>
<td>$\widehat{QZ}<em>k(n) := \text{End}</em>{A_n}(R_n^{\otimes k})$</td>
<td>$\widehat{QZ}<em>{k,n}^\lambda$, $\lambda \vdash n$, $\lambda &gt; \lambda^*$, $\lambda \in q\Lambda</em>{k,A_n} \subseteq \Lambda_{A_n}$</td>
</tr>
<tr>
<td>$\widehat{QZ}<em>{k+\frac{1}{2}}(n) := \text{End}</em>{A_{n-1}}(R_n^{\otimes k})$</td>
<td>$\widehat{QZ}<em>{k+\frac{1}{2},n}^\mu$, $\mu \vdash n - 1$, $\mu &gt; \mu^*$, $\mu \in q\Lambda</em>{k+\frac{1}{2},A_{n-1}} \subseteq \Lambda_{A_{n-1}}$</td>
</tr>
</tbody>
</table>

Table 2: Notation for the centralizer algebras and modules associated with the tensor product $R_n^{\otimes k}$ of the reflection module $R_n = S_n^{[n-1,1]}$ of $S_n$ and its restriction to $S_{n-1}, A_n$, and $A_{n-1}$.

The permutation module $M_n$ of the symmetric group satisfies $M_n \cong R_n \oplus S_n^{[n]}$, where $R_n = S_n^{[n-1,1]}$ is the $(n-1)$-dimensional reflection representation of $S_n$ and $S_n^{[n]}$ is the trivial module. Applying Proposition 2.11(a) and Theorem 6.1 gives the following:
Theorem 7.1. Let \( k, n \in \mathbb{Z}_{\geq 0} \) with \( n \geq 1 \). The dimensions of the irreducible modules for \( QZ_k(n) \), \( QZ_{k+\frac{1}{2}}(n) \), \( QZ_k(n) \) and \( QZ_{k+\frac{1}{2}}(n) \) are as follows (using notation from Tables \( 1 \) and \( 2 \)).

(a) For \( \lambda \vdash n \), \( \lambda \in qA_k.S_n \),
\[
\dim QZ^\lambda_{k,n} = \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} \dim Z^\lambda_{\ell,n}.
\]

(b) For \( \mu \vdash n - 1 \), \( \mu \in qA_k.S_{n-1} \),
\[
\dim QZ^\mu_{k+\frac{1}{2},n} = \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} \dim Z^\mu_{\ell+\frac{1}{2},n}.
\]

(c) For \( \lambda \vdash n \) with \( \lambda \in qA_k.A_n \),
\[
\dim \tilde{QZ}_{k,n}^\lambda = \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} \dim \tilde{Z}_{k,n}^\lambda = \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} \left( \dim Z_{\ell,n}^\lambda + \dim Z_{\ell+\frac{1}{2},n}^\lambda \right), \text{ if } \lambda > \lambda^*,
\]
\[
\dim \tilde{QZ}_{k,n}^{\lambda^\pm} = \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} \dim \tilde{Z}_{k,n}^{\lambda^\pm} = \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} \dim Z_{\ell,n}^{\lambda^\pm} = \dim QZ_{k,n}^\lambda, \text{ if } \lambda = \lambda^*.
\]

(d) For \( \mu \vdash n - 1 \) with \( \mu \in qA_{k+\frac{1}{2}}.A_{n-1} \),
\[
\dim \tilde{QZ}_{k+\frac{1}{2},n}^\mu = \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} \dim Z_{\ell+\frac{1}{2},n}^\mu
\]
\[
= \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} \left( \dim Z_{\ell+\frac{1}{2},n}^\mu + \dim Z_{\ell+\frac{1}{2},n}^{\mu^*} \right), \text{ if } \mu > \mu^*,
\]
\[
\dim \tilde{QZ}_{k+\frac{1}{2},n}^{\mu^\pm} = \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} \dim Z_{\ell+\frac{1}{2},n}^{\mu^\pm} = \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} \dim Z_{\ell+\frac{1}{2},n}^{\mu^*} = \dim QZ_{k+\frac{1}{2},n}^\mu, \text{ if } \mu = \mu^*.
\]

Applying Proposition 2.11(b) and Theorem 7.1 gives the following:

Corollary 7.2. Let \( k, n \in \mathbb{Z}_{\geq 0} \) with \( n > 0 \), and let the notation be as in Table 2.

(a) \( \dim QZ_k(n) = \dim QZ_{2k,n}^{[n]} = \sum_{\ell=0}^{2k} (-1)^{2k-\ell} \binom{2k}{\ell} B(\ell, n) \)
\[
= \sum_{\ell=0}^{2k} (-1)^{2k-\ell} \binom{2k}{\ell} B(\ell) = 1 + \sum_{\ell=1}^{2k} (-1)^{\ell-1} B(2k - \ell) \text{ if } n \geq 2k + 2.
\]

(b) \( \dim QZ_{k+\frac{1}{2}}(n) = \dim QZ_{2k+\frac{1}{2},n}^{[n-1]} = \sum_{\ell=0}^{2k} (-1)^{2k-\ell} \binom{2k}{\ell} B(\ell + 1, n) \)
\[
= \sum_{\ell=0}^{2k} (-1)^{2k-\ell} \binom{2k}{\ell} B(\ell + 1) = B(2k) \text{ if } n \geq 2k + 1.
\]
Let \eqref{5.30}, \eqref{5.31}, and Corollary 6.2 (b) and (d). The final equality in (a) and (c) can be seen as follows:

The first two equalities in parts (a)-(d) of the corollary follow from Proposition 2.11 (b),

\begin{proof}
The first two equalities in parts (a)-(d) of the corollary follow from Proposition 2.11 (b),

Remark 7.3. The result from Corollary 7.2 (a) that \( \dim \mathcal{Q}Z_k(n) = 1 + \sum_{\ell=1}^{2k} (-1)^{\ell-1} B(2k-\ell) \) when \( n \geq 2k + 2 \) was shown in \cite{DO} Cor. 2.6. As noted there, the sequence \( \{v_\ell\} \) is \#A000296 in \cite{OEIS}.

The results in Theorem 7.1 enable us to conclude the following for generic quasi-partition algebras.

Corollary 7.4. Let \( \nu \) be a partition with \( 0 \leq |\nu| \leq k \). For \( \xi \notin \{0, 1, \ldots, 2k - 1\} \), let \( \mathcal{Q}P_{k,\xi} \) denote the irreducible \( \mathcal{Q}P_k(\xi) \)-module. Then

\[
\dim \mathcal{Q}P_{k,\xi} = f^\nu \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} \sum_{t=|\nu|}^{\ell} \binom{\ell}{t} \binom{t}{|\nu|} \quad (= f^\nu \text{ when } |\nu| = k).
\]

The Bratteli diagrams constructed using the reflection module \( R_n \) for the pairs \( (S_n, S_{n-1}) \), \( (A_n, A_{n-1}) \) for \( n = 6 \) are displayed in \cite{A2} and \cite{A4} of the Appendix. The subscript on a partition at level \( \ell \in \frac{1}{2} \mathbb{Z}_{\geq 0} \) is the dimension of the irreducible module for the centralizer algebra \( \mathcal{Q}Z_\ell(6) \). For \( k \in \mathbb{Z}_{\geq 0} \), \( \text{Ind}^S_{S_{n-1}} \mathbb{R} \mathbb{S}_{S_{n-1}}^S (R_n^{\otimes k}) \) is isomorphic as an \( S_n \)-module to \( \mathbb{R} (R_n^{\otimes k} \oplus R_n^{\otimes (k+1)}) \). This implies that the subtrees on level \( k + \frac{1}{2} \) are gotten from level \( k \) by Pascal addition; however, the subtrees on level \( k + 1 \) are obtained by first performing Pascal addition from level \( k + \frac{1}{2} \) and then subtracting the corresponding subscript from level \( k \).
References


[D] S. Ding, Tensor powers of the defining representation of $S_n$, arXiv #1508.05433.


A Appendix: Bratteli Diagrams

A.1 Levels $\ell = 0, \frac{1}{2}, 1, \ldots, \frac{7}{2}, 4$ of the Bratteli diagram $\mathcal{B}(S_6, S_5)$

Level $\ell = \frac{7}{2}$ is the first time the centralizer algebra loses a dimension from the generic dimension, which is the 7th Bell number $B(7) = 877$. 

![Bratteli Diagram](image-url)
A.2 Levels $\ell = 0, \frac{1}{2}, 1, \ldots, \frac{7}{2}, 4$ of the quasi-Bratteli diagram $QB(S_6, S_5)$

To calculate the subscripts on the half-integer rows, use Pascal addition of the subscripts from the row above. To calculate the subscripts on integer level rows, first use Pascal addition from the row above, and then subtract the subscript on the same partition from two rows above.
A.3 Levels $\ell = 0, \frac{1}{2}, 1, \ldots, \frac{7}{2}, 4$ of the Bratteli diagram $\mathcal{B}(A_6, A_5)$
A.4 Levels $\ell = 0, \frac{1}{2}, 1, \ldots, \frac{7}{2}, 4$ of the quasi-Bratteli diagram $\Omega B(A_6, A_5)$

To calculate the subscripts on the half-integer rows, use Pascal addition of the subscripts from the row above. To calculate the subscripts on integer level rows, first use Pascal addition from the row above, and then subtract the subscript on the same partition from two rows above.
B Appendix: Proposition 5.12(b), Proof and Examples

This appendix is devoted to a proof and examples of Proposition 5.12(b), which says the following:

**Proposition 5.12(b)** Suppose \( \lambda \) is a partition of \( n \) and \( 1 \leq r \leq \ell(\lambda) \). If \( \lambda_r > \lambda_{r+1} \), and \( \lambda_r > r+1 \), then \( \mathcal{F}_r^\lambda = \mathcal{F}_{r-\square}^\lambda \).

**Proof.** Since \( \lambda_r > r+1 \), we have \((\lambda - \square_r)_r > r\), so both \( \lambda \) and \( \lambda - \square_r \) satisfy the requirements of Definition 5.5. Let \( \mathcal{F}_r^\lambda \) be the set of pairs \((\pi, T)\) such that \( \pi \) is an \( r \)-sequence for \( \lambda \) and \( T \) is a standard Young tableau of shape \( \lambda \) that is compatible with \( \pi \) as in Remark 5.10. Then \( \pi = \{\pi(0) = [\lambda_r, 0, \ldots, 0], \pi(1), \ldots, \pi(r-1), \pi(r) = \lambda_r\} \), where

\[
\pi_1^{(1)} = \pi_2^{(1)} = \pi_3^{(2)} = \ldots = \pi_{r-1}^{(r-1)} = \pi_r^{(r-1)} = \lambda_r,
\]

and \( \mathcal{F}_r^\lambda = |\mathcal{F}_r^\lambda| \). We prove the proposition by demonstrating a bijection between \( \mathcal{F}_r^\lambda \) and \( \mathcal{F}_{r-\square}^\lambda \).

\((\mathcal{F}_r^\lambda \rightarrow \mathcal{F}_{r-\square}^\lambda)\). Given a pair \((\pi, T) \in \mathcal{F}_r^\lambda\), apply the following algorithm.

1. Remove the entry \( \lambda_r \) in position \((1, \lambda_r)\) of \( T \).
2. For \( i = 2, 3, \ldots, r \), (in that order) do the following:
   a. Let \( a_i \) be the entry in position \((i, \lambda_r)\) of \( T \), and let \( b_i = |\pi(i-1)| \).
   b. Promote the entries \( a_i, a_i + 1, \ldots, b_i \) from \( \pi(i-1)/\pi(i-2) \) to \( \pi(i)/\pi(i-1) \).
   c. Remove entry \( a_i \) from position \((i, \lambda_r)\) in \( T \).
   d. Reduce the values of entries \( a_i + 1, \ldots, b_i \) by 1.
   e. Place \( b_i \) in position \((i-1, \lambda_r)\) in \( T \).
3. Decrease the values of the entries \( \lambda_r + 1, \lambda_r + 2, \ldots, n \) by 1, and remove the empty box at position \((r, \lambda_r)\).

To begin, we remove entry \( \lambda_r \) from the box at position \((1, \lambda_r)\) and set \( \pi'(0) = \pi(0) - \square_1 = [\lambda_r - 1, 0, \ldots, 0] \). Next observe for \( i \geq 2 \) that since \( a_i \) is the entry in box \((i, \lambda_r)\) of \( T \), and \( \pi(i-1) = \lambda_r \), the entries \( a_i, a_i + 1, \ldots, b_i = |\pi(i-1)| \) all lie in the skew partition \( \pi(i-1)/\pi(i-2) \) for \( i = 2 \). This is clear since \( a_2 > \lambda_r \). For \( i > 2 \), we have \( a_i > a_{i-1} \) because column \( \lambda_r \) of \( T \) is standard, and since \( a_{i-1} \) has been moved to \( \pi(i-1)/\pi(i-2) \) in the previous step, \( a_i \notin \pi(i-2) \). Now promoting the entries \( a_i, a_i + 1, \ldots, b_i \) from \( \pi(i-1)/\pi(i-2) \) to \( \pi(i)/\pi(i-1) \), removing \( a_i \), and then decreasing each of the entries \( a_i + 1, \ldots, b_i \) by 1 leaves a vacancy at box \((i, \lambda_r)\) while maintaining standardness, since the entries in the skew shape \( \pi(j)/\pi(i-1) \) for \( j > i-1 \) are all larger than \( b_i \). Because \( b_i \leq b_{i+1} \), the algorithm guarantees that the entries in column \( \lambda_r \) are increasing. If there is a box \((i-1, \lambda_r + 1)\), it cannot lie in \( \pi(i-1) \), since \( \pi(i-1) = \lambda_r \), by the \( r \)-sequence property. Therefore, at step \( i \), any entry in box \((i-1, \lambda_r + 1)\) must lie in some \( \pi(j)/\pi(i-1) \) for \( j > i-1 \), and hence must be larger than \( b_i \). Thus, placing \( b_i \) in box \((i-1, \lambda_r)\) preserves standardness. At the end of step \( i \), define \( \pi(i-1) \) to be the partition formed by the boxes containing the entries \( < a_i \), but note that the entry \( \lambda_r \) is missing. Box \((i, \lambda_r)\) is vacant, and if \( i < r \), it is filled in the next step by \( b_{i+1} \geq a_{i+1} > a_i \), so \( \pi(i-1) = \lambda_r - 1 \). Because \( b_i \) is in box \((i-1, \lambda_r)\), and \( b_i \geq a_i \), it must be that \( \pi(i-1) \leq \lambda_r - 1 \). However, \( \pi(i-1) \geq \pi(i-1) = \lambda_r - 1 \), so these two relations combine to show \( \pi(i-1) = \pi(i-1) = \lambda_r - 1 \). At the completion of part (2) of the algorithm, we have defined partitions \( \pi'(0) \subseteq \pi'(1) \subseteq \ldots \subseteq \pi'(r-1) \) satisfying \( \pi(i-1) = \pi(i-1) = \lambda_r - 1 \) for \( i = 2, 3, \ldots, r \). The tableau
$T'$ resulting from the algorithm has shape $\lambda - \square_r$, because box $(r, \lambda_r)$ was removed in part (3). Moreover, $T'$ is standard, since each step $i = 2, 3, \ldots, r$ maintains the standard property, and $T'$ is compatible with the $r$-sequence $\pi'_r = \{\pi'_r(0), \pi'_r(1), \ldots, \pi'_r(r-1), \pi'_r(r) := \lambda - \square_r\}$ by construction. Therefore, the algorithm starts with $(\bar{\pi}, T) \in \mathcal{F}_r^\lambda$ and ends with $(\bar{\pi}', T') \in \mathcal{F}_r^{\lambda - \square_r}$.

We show that the above algorithm is a bijection by arguing that the following algorithm is its inverse.

\[(3)^r \leftarrow \mathcal{F}_r^{\lambda - \square_r}. \text{ Given a pair } (\bar{\pi}, U) \in \mathcal{F}_r^{\lambda - \square_r}, \text{ apply the following algorithm.} \]

1') Increase the values of the entries $\lambda_r, \lambda_r + 1, \ldots, n - 1$ by 1, and add an empty box at position $(r, \lambda_r)$.

2') For $i = r, r - 1, \ldots, 2$, (in that order) do the following:
   a) Let $b_i$ be the entry in position $(i - 1, \lambda_r)$ of $U$, and let $a_i = |g(i-1)| + 2$.
   b) Remove $b_i$ from position $(i - 1, \lambda_r)$.
   c) Increase the values of entries $a_i, \ldots, b_i - 1$ by 1.
   d) Place $a_i$ in position $(i, \lambda_r)$.
   e) Demote entries $a_i, a_i + 1, \ldots, b_i$ from $g(i) / g(i-1)$ to $g(i-1) / g(i-2)$.

3') Add the entry $\lambda_r$ in position $(1, \lambda_r)$ in $U$.

These algorithms are inverses, since they undo one another (trivially) step-by-step in reverse order: $(3) \leftrightarrow (1)', (2e) \leftrightarrow (2'b), (2d) \leftrightarrow (2'c), (2c) \leftrightarrow (2'd), (2b) \leftrightarrow (2'e), (1) \leftrightarrow (3)'$. Furthermore, the algorithms are stated in such a way that the values of $a_i$ and $b_i$ are the same in either direction. All that remains to prove is that the inverse algorithm maps an element of $\mathcal{F}_r^{\lambda - \square_r}$ to $\mathcal{F}_r^\lambda$. This argument proceeds as with the first algorithm. At the end of $(1)'$, set $g'(r) = \lambda$ because a box has been added to $\lambda - \square_r$ at position $(r, \lambda_r)$. After performing step $i$ of part $(2)'$, let $g'(i-1)$ be the partition formed by the empty box at position $(i - 1, \lambda_r)$ along with the boxes containing the entries $\leq b_i$ for $i = r, r - 1, \ldots, 2$ (in that order). Let $U'$ be the resulting tableau and $g'$ be the resulting sequence. Then $U'$ is standard because at each step the values are increased relative to one another. At step $i$, $a_i$ can be placed in position $(i, \lambda_r)$ since $a_i = |g(i-1)| + 2$ (we add 2 here instead of 1 because in step $(1)'$ the values of $U$ have been increased by 1). We argue that the final sequence $g'$ satisfies $g'(i-1) = g'(i) = \lambda_r$ as follows. We have $a_i$ in position $(i, \lambda_r)$ and $a_i \leq b_i$, so $a_i$ is in $g'(i-1)$, and consequently, $g'(i-1) \geq g'(i) \geq \lambda_r$. As $b_i$ was in the box in position $(i - 1, \lambda_r)$, everything to the right of that position is in $g'(j)$ for some $j > i - 1$. Therefore, $\lambda_r \geq g'(i-1) \geq g'(i) \geq \lambda_r$. Thus, $g' = \{g'(0) := [\lambda_r, 0, \ldots, 0], g'(1), \ldots, g'(r) = \lambda\}$ is an $r$-sequence for $\lambda$. The tableau $U'$ resulting from the algorithm has shape $\lambda$, because box $(r, \lambda_r)$ was added in part $(1)'$. Moreover, $U'$ is standard, since each step $i = 2, 3, \ldots, r$ maintains the standard property, and $U'$ is compatible with the $r$-sequence $g'$ by construction. Therefore, the algorithm starts with $(\bar{\pi}, U) \in \mathcal{F}_r^{\lambda - \square_r}$ and ends with $(g', U') \in \mathcal{F}_r^\lambda$. \hfill $\square$

To follow are four examples of the algorithms used in the proof of Proposition 5.12(b).
Example B.1. This example demonstrates the bijection \( \mathcal{F}_2^{[5,4,1]} \leftrightarrow \mathcal{F}_2^{[5,3,1]} \). The leftmost column contains the 6 standard tableaux compatible with a 2-sequence for \( \lambda = [5, 4, 1] \) (c.f., Example 5.9). Here, \( \lambda_r = \lambda_2 = 4 \). For (a) and (b) the corresponding 2-sequence is \( \pi = \{[4, 0, 0], [4, 4, 0], [5, 4, 1]\} \). For (c), (d), (e) the 2-sequence is \( \pi = \{[4, 0, 0], [4, 4, 1], [5, 4, 1]\} \).

The rightmost column contains the 6 standard tableaux compatible with a 2-sequence for \( \lambda - \emptyset_2 = [5, 3, 1] \). Here \( (\lambda - \emptyset_2)_r = (\lambda - \emptyset_2)_2 = 3 \). For (a), (b), (c) the corresponding 2-sequence is \( \underline{\pi} = \{[3, 0, 0], [3, 3, 0], [5, 3, 1]\} \). For (d) and (e) the 2-sequence is \( \underline{\pi} = \{[3, 0, 0], [3, 3, 1], [5, 3, 1]\} \).

This example illustrates that two tableaux compatible with the same underlying \( r \)-sequence do not necessarily map to tableaux with the same underlying \( r \)-sequence.

\[
\begin{array}{cccccccccc}
\text{a)} & a_2 = 8, b_2 = 8 & (1),(26) & a_2 = 8, b_2 = 8 & (2cde) & a_2 = 8, b_2 = 8 & (3) & 12379 \leftarrow 4569 \\
& 123410 & \leftrightarrow & 12310 & \leftrightarrow & 123810 & \leftrightarrow & 123794569 \\
& 5678 & \leftrightarrow & 5678 & \leftrightarrow & 567 & \leftrightarrow & 4569 \\
& 9 & \leftrightarrow & 9 & \leftrightarrow & 9 & \leftrightarrow & 8 \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
\text{b)} & a_2 = 8, b_2 = 8 & (1),(26) & a_2 = 8, b_2 = 8 & (2cde) & a_2 = 8, b_2 = 8 & (3) & 12378 \leftarrow 4569 \\
& 12349 & \leftrightarrow & 1239 & \leftrightarrow & 12389 & \leftrightarrow & 123784569 \\
& 5678 & \leftrightarrow & 5678 & \leftrightarrow & 567 & \leftrightarrow & 4569 \\
& 10 & \leftrightarrow & 10 & \leftrightarrow & 10 & \leftrightarrow & 9 \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
\text{c)} & a_2 = 8, b_2 = 9 & (1),(26) & a_2 = 8, b_2 = 9 & (2cde) & a_2 = 8, b_2 = 9 & (3) & 12389 \leftarrow 4567 \\
& 123410 & \leftrightarrow & 12310 & \leftrightarrow & 123910 & \leftrightarrow & 123894567 \\
& 5678 & \leftrightarrow & 5678 & \leftrightarrow & 567 & \leftrightarrow & 4567 \\
& 9 & \leftrightarrow & 9 & \leftrightarrow & 9 & \leftrightarrow & 7 \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
\text{d)} & a_2 = 9, b_2 = 9 & (1),(26) & a_2 = 9, b_2 = 9 & (2cde) & a_2 = 9, b_2 = 9 & (3) & 12389 \leftarrow 4567 \\
& 12349 & \leftrightarrow & 1239 & \leftrightarrow & 123910 & \leftrightarrow & 123894567 \\
& 5679 & \leftrightarrow & 5679 & \leftrightarrow & 567 & \leftrightarrow & 4567 \\
& 8 & \leftrightarrow & 8 & \leftrightarrow & 8 & \leftrightarrow & 7 \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
\text{e)} & a_2 = 9, b_2 = 9 & (1),(26) & a_2 = 9, b_2 = 9 & (2cde) & a_2 = 9, b_2 = 9 & (3) & 12389 \leftarrow 4567 \\
& 123410 & \leftrightarrow & 12310 & \leftrightarrow & 123910 & \leftrightarrow & 123894567 \\
& 5689 & \leftrightarrow & 5689 & \leftrightarrow & 568 & \leftrightarrow & 4567 \\
& 7 & \leftrightarrow & 7 & \leftrightarrow & 7 & \leftrightarrow & 6 \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
\text{f)} & a_2 = 9, b_2 = 9 & (1),(26) & a_2 = 9, b_2 = 9 & (2cde) & a_2 = 9, b_2 = 9 & (3) & 12389 \leftarrow 4567 \\
& 123410 & \leftrightarrow & 12310 & \leftrightarrow & 123910 & \leftrightarrow & 123894567 \\
& 5789 & \leftrightarrow & 5789 & \leftrightarrow & 578 & \leftrightarrow & 4567 \\
& 6 & \leftrightarrow & 6 & \leftrightarrow & 6 & \leftrightarrow & 5 \\
\end{array}
\]

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**Example B.2.** Assume $\lambda = [8, 7, 5, 4, 3] \vdash 27$ and $r = 3$, so that $\lambda_r = \lambda_3 = 5$. Then the standard tableau $T$ below is a $\pi$-compatible for the 3-sequence for $\lambda$ given by

$$\pi = \{[5, 0, 0, 0, 0], [5, 5, 4, 2, 0], [6, 5, 5, 4, 2], [8, 7, 5, 4, 3]\}.$$ 

$$T = \begin{array}{c|c|c|c|c|c|c|c|c} 1 & 2 & 3 & 4 & 5 & 22 & 23 & 24 \\ 6 & 8 & 10 & 11 & 14 & 26 & 27 \\ 7 & 12 & 15 & 16 & 17 \\ 9 & 13 & 18 & 20 \\ 19 & 21 & 25 \\ \end{array} \quad \begin{array}{c|c|c|c|c|c|c|c|c} 1 & 2 & 3 & 4 & 22 & 23 & 24 \\ 6 & 8 & 10 & 11 & 14 & 26 & 27 \\ 7 & 12 & 15 & 16 & 17 \\ 9 & 13 & 18 & 20 \\ 19 & 21 & 25 \\ \end{array} \quad \begin{array}{c|c|c|c|c|c|c|c|c} 1 & 2 & 3 & 4 & 22 & 23 & 24 \\ 6 & 8 & 10 & 11 & 14 & 26 & 27 \\ 7 & 12 & 15 & 16 & 17 \\ 9 & 13 & 18 & 20 \\ 19 & 21 & 25 \\ \end{array}$$

$i = 2$

$$a_2 = 14, b_2 = 16$$

$$\begin{array}{c|c|c|c|c|c|c|c|c} 1 & 2 & 3 & 4 & 22 & 23 & 24 \\ 6 & 8 & 10 & 11 & 14 & 26 & 27 \\ 7 & 12 & 15 & 16 & 17 \\ 9 & 13 & 18 & 20 \\ 19 & 21 & 25 \\ \end{array} \quad \begin{array}{c|c|c|c|c|c|c|c|c} 1 & 2 & 3 & 4 & 22 & 23 & 24 \\ 6 & 8 & 10 & 11 & 14 & 26 & 27 \\ 7 & 12 & 15 & 16 & 17 \\ 9 & 13 & 18 & 20 \\ 19 & 21 & 25 \\ \end{array} \quad \begin{array}{c|c|c|c|c|c|c|c|c} 1 & 2 & 3 & 4 & 22 & 23 & 24 \\ 6 & 8 & 10 & 11 & 14 & 26 & 27 \\ 7 & 12 & 15 & 16 & 17 \\ 9 & 13 & 18 & 20 \\ 19 & 21 & 25 \\ \end{array}$$

$$a_2 = 14, b_2 = 16$$

$$\begin{array}{c|c|c|c|c|c|c|c|c} 1 & 2 & 3 & 4 & 22 & 23 & 24 \\ 6 & 8 & 10 & 11 & 14 & 26 & 27 \\ 7 & 12 & 15 & 16 & 17 \\ 9 & 13 & 18 & 20 \\ 19 & 21 & 25 \\ \end{array} \quad \begin{array}{c|c|c|c|c|c|c|c|c} 1 & 2 & 3 & 4 & 22 & 23 & 24 \\ 6 & 8 & 10 & 11 & 14 & 26 & 27 \\ 7 & 12 & 15 & 16 & 17 \\ 9 & 13 & 18 & 20 \\ 19 & 21 & 25 \\ \end{array} \quad \begin{array}{c|c|c|c|c|c|c|c|c} 1 & 2 & 3 & 4 & 22 & 23 & 24 \\ 6 & 8 & 10 & 11 & 14 & 26 & 27 \\ 7 & 12 & 15 & 16 & 17 \\ 9 & 13 & 18 & 20 \\ 19 & 21 & 25 \\ \end{array}$$

$i = 3$

$$a_3 = 17, b_3 = 22$$

$$\begin{array}{c|c|c|c|c|c|c|c|c} 1 & 2 & 3 & 4 & 16 & 22 & 23 & 24 \\ 6 & 8 & 10 & 11 & 14 & 26 & 27 \\ 7 & 12 & 14 & 15 & 17 \\ 9 & 13 & 18 & 20 \\ 19 & 21 & 25 \\ \end{array} \quad \begin{array}{c|c|c|c|c|c|c|c|c} 1 & 2 & 3 & 4 & 16 & 22 & 23 & 24 \\ 6 & 8 & 10 & 11 & 14 & 26 & 27 \\ 7 & 12 & 14 & 15 & 17 \\ 9 & 13 & 18 & 20 \\ 19 & 21 & 25 \\ \end{array} \quad \begin{array}{c|c|c|c|c|c|c|c|c} 1 & 2 & 3 & 4 & 16 & 22 & 23 & 24 \\ 6 & 8 & 10 & 11 & 14 & 26 & 27 \\ 7 & 12 & 14 & 15 & 17 \\ 9 & 13 & 18 & 20 \\ 19 & 21 & 25 \\ \end{array}$$

$$a_3 = 17, b_3 = 22$$

$$\begin{array}{c|c|c|c|c|c|c|c|c} 1 & 2 & 3 & 4 & 16 & 22 & 23 & 24 \\ 6 & 8 & 10 & 11 & 14 & 26 & 27 \\ 7 & 12 & 14 & 15 & 17 \\ 9 & 13 & 17 & 19 \\ 18 & 20 & 25 \\ \end{array} \quad \begin{array}{c|c|c|c|c|c|c|c|c} 1 & 2 & 3 & 4 & 16 & 22 & 23 & 24 \\ 6 & 8 & 10 & 11 & 14 & 26 & 27 \\ 7 & 12 & 14 & 15 & 17 \\ 9 & 13 & 17 & 19 \\ 18 & 20 & 25 \\ \end{array} \quad \begin{array}{c|c|c|c|c|c|c|c|c} 1 & 2 & 3 & 4 & 16 & 22 & 23 & 24 \\ 6 & 8 & 10 & 11 & 14 & 26 & 27 \\ 7 & 12 & 14 & 15 & 17 \\ 9 & 13 & 17 & 19 \\ 18 & 20 & 25 \\ \end{array}$$

The final tableau $T'$ is compatible with the 3-sequence $\pi'$ for $\lambda - \square_r$ given by

$$\pi' = \{[4, 0, 0, 0, 0], [4, 4, 2, 2, 0], [5, 4, 2, 2, 0], [8, 7, 4, 4, 3]\}.$$
Example B.3. Assume $\lambda = [8, 7, 5, 4, 3] \vdash 27$ and $r = 3$, so that $\lambda_r = \lambda_3 = 5$. Then the standard tableau $T$ below is compatible with the 3-sequence $\pi$ for $\lambda$ given by

$$\pi = \{ [5, 0, 0, 0, 0], [5, 5, 5, 2, 0], [5, 5, 5, 4, 2], [8, 7, 5, 4, 3] \} .$$

$$T = \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & \text{22 23 24} \\
6 & 8 & 10 & 11 & 14 & 26 27 \\
7 & 12 & 15 & 16 & 17 \\
9 & 13 & 18 & 20 & \text{19 21 25}
\end{array}
\begin{array}{cccccc}
1 & 2 & 3 & 4 & \text{22 23 24} \\
6 & 8 & 10 & 11 & 14 & 26 27 \\
7 & 12 & 15 & 16 & 17 \\
9 & 13 & 18 & 20 & \text{19 21 25}
\end{array}
\begin{array}{cccccc}
(1) & (3) & = (3') \\
\end{array}$$

$i = 2$

$$a_2 = 14, b_2 = 17$$

$$\begin{array}{cccccc}
1 & 2 & 3 & 4 & \text{22 23 24} \\
6 & 8 & 10 & 11 & 14 & 26 27 \\
7 & 12 & 15 & 16 & 17 \\
9 & 13 & 18 & 20 & \text{19 21 25}
\end{array}
\begin{array}{cccccc}
1 & 2 & 3 & 4 & \text{22 23 24} \\
6 & 8 & 10 & 11 & 14 & 26 27 \\
7 & 12 & 15 & 16 & 17 \\
9 & 13 & 18 & 20 & \text{19 21 25}
\end{array}
\begin{array}{cccccc}
(2b) & (2'c) & \leftarrow (2d) & \leftarrow (2'e) \\
\end{array}$$

$$\begin{array}{cccccc}
1 & 2 & 3 & 4 & \text{22 23 24} \\
6 & 8 & 10 & 11 & 14 & 26 27 \\
7 & 12 & 15 & 16 & 17 \\
9 & 13 & 18 & 20 & \text{19 21 25}
\end{array}
\begin{array}{cccccc}
1 & 2 & 3 & 4 & \text{22 23 24} \\
6 & 8 & 10 & 11 & 14 & 26 27 \\
7 & 12 & 15 & 16 & 17 \\
9 & 13 & 18 & 20 & \text{19 21 25}
\end{array}
\begin{array}{cccccc}
(2b) & (2'c) & \leftarrow (2d) & \leftarrow (2'e) \\
\end{array}$$

$i = 3$

$$a_3 = 16, b_3 = 21$$

$$\begin{array}{cccccc}
1 & 2 & 3 & 4 & 17 & \text{22 23 24} \\
6 & 8 & 10 & 11 & 26 27 \\
7 & 12 & 14 & 15 & 16 \\
9 & 13 & 18 & 20 & \text{19 21 25}
\end{array}
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 17 & \text{22 23 24} \\
6 & 8 & 10 & 11 & 26 27 \\
7 & 12 & 14 & 15 & 16 \\
9 & 13 & 18 & 20 & \text{19 21 25}
\end{array}
\begin{array}{cccccc}
(2b) & (2'c) & \leftarrow (2d) & \leftarrow (2'e) \\
\end{array}$$

$$\begin{array}{cccccc}
1 & 2 & 3 & 4 & 16 & \text{22 23 24} \\
6 & 8 & 10 & 11 & 26 27 \\
7 & 12 & 14 & 15 & 16 \\
9 & 13 & 17 & 19 & \text{18 20 25}
\end{array}
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 16 & \text{22 23 24} \\
6 & 8 & 10 & 11 & 26 27 \\
7 & 12 & 14 & 15 & 16 \\
9 & 13 & 17 & 19 & \text{18 20 25}
\end{array}
\begin{array}{cccccc}
(2b) & (2'c) & \leftarrow (2d) & \leftarrow (2'e) \\
\end{array}$$

The final tableau $T'$ is compatible with the 3-sequence $\pi'$ for $\lambda - \square$, given by

$$\{ [4, 0, 0, 0, 0], [4, 4, 2, 2, 0], [4, 4, 4, 2, 0], [8, 7, 4, 4, 3] \} .$$
Example B.4. Assume $\lambda = [6, 6, 6, 6]$ and $r = 4$, so that $\lambda_r = \lambda_4 = 6$. The initial tableau $T$ below is $\pi$-compatible for the 4-sequence $\pi$ of $\lambda$ given by

$$\pi = \{[6, 0, 0, 0], [6, 6, 2, 0], [6, 6, 6, 1], [6, 6, 6, 6], [6, 6, 6, 6]\}.$$ 

The final tableau is $\pi'$-compatible for the 4-sequence $\pi'$ given by

$$\pi' = \{[5, 0, 0, 0], [5, 5, 0, 0], [6, 5, 5, 0], [6, 6, 5, 5], [6, 6, 6, 5]\}.$$ 

Example B.5. Let $\lambda = [6, 6, 6, 6], r = 4, \pi = \{[6, 0, 0, 0], [6, 6, 6, 6], [6, 6, 6, 6], [6, 6, 6, 6], [6, 6, 6, 6]\}$.

The final tableau is $\pi'$-compatible for the 4-sequence $\pi'$ given by

$$\pi' = \{[5, 0, 0, 0], [5, 5, 0, 0], [6, 5, 5, 0], [5, 5, 5, 5], [6, 6, 6, 5]\}.$$
C Appendix: Values of $F^\lambda_r$ for all partitions $\lambda$ of $n$ with $n \leq 12$

The tables below display all the values $F^\lambda_r$ for all partitions $\lambda \vdash n \leq 12$ such that $F^\lambda_r$ is nonzero for some value of $r$. All such $r$ are given. Mathematica code for computing $F^\lambda_r$ and the dimension formulas of this paper can be found on the second author’s web page: [link]

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