Recasting the $2p_{\pm 1}$ and $2p_{-1}$ orbitals into a more useful form:

$$\Psi_{2p_{\pm 1}} = R_{21}(r) Y_{1, \pm 1}(\theta, \phi)$$

$$= \left[ \frac{1}{4\sqrt{6}} \left( \frac{Z}{a_0} \right)^{3/2} \frac{(2Zr)}{a_0} \right] e^{-\frac{Zr}{2a_0}} \left[ -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi} \right]$$

$$= \left[ \frac{1}{4\sqrt{6}} \left( \frac{Z}{a_0} \right)^{3/2} \frac{(2Z)}{a_0} \right] \left( -\sqrt{\frac{3}{8\pi}} \right) e^{-\frac{Zr}{2a_0}} r \sin \theta e^{i\phi}$$

$$\Psi_{2p_{\pm 1}} = -f(r) r \sin \theta e^{i\phi}$$

Similarly, $\Psi_{2p_{-1}} = R_{21}(r) Y_{1, -1}(\theta, \phi) = [f(r)] r \sin \theta e^{-i\phi}$

As already proven, I am free to take any linear combination of energy eigenfunctions I want, with the guarantee that this linear combination will also be an energy eigenfunction (i.e., it will satisfy the Schrödinger equation).

Let me choose

$$\Psi = -\frac{1}{\sqrt{2}} \Psi_{2p_{+1}} + \frac{1}{\sqrt{2}} \Psi_{2p_{-1}}$$

Choosing $C_1 = -\frac{1}{\sqrt{2}}$ and $C_2 = +\frac{1}{\sqrt{2}}$ will

(a) ensure my new $\Psi$ is normalized
(b) get rid of the imaginary part of the spherical harmonics
\[ \Psi = \frac{1}{\sqrt{2}} [f(r)] \sin \theta e^{i \phi} + \frac{1}{\sqrt{2}} [f(r)] \sin \theta e^{-i \phi} \]

\[ \Psi = \frac{1}{\sqrt{2}} [f(r)] \sin \theta [e^{i \phi} + e^{-i \phi}] \]

\[ \text{and since} \quad e^{i \phi} = \cos \phi + i \sin \phi \]
\[ \text{and} \quad e^{-i \phi} = \cos \phi - i \sin \phi, \]

\[ e^{i \phi} + e^{-i \phi} = 2 \cos \phi \]

\[ \Psi = \frac{2}{\sqrt{2}} [f(r)] \sin \theta \cos \phi \]

and what is this in Cartesian coordinates?

\[ \cos \phi = \frac{x}{r \sin \theta} \]
\[ x = r \sin \theta \cos \phi \]

so \[ \Psi = -\frac{1}{\sqrt{2}} \Psi_{2p+1} + \frac{1}{\sqrt{2}} \Psi_{2p-1} = \sqrt{2} [f(r)] X \]

and \[ \Psi = 0 \text{ when } x = 0. \] This eigenfunction has an angular node in the \( yz \) plane, and is concentrated on the \( x \)-axis.

\[ -\frac{1}{\sqrt{2}} \Psi_{2p+1} + \frac{1}{\sqrt{2}} \Psi_{2p-1} = \Psi_{2p_x} \quad (\text{node at } x = 0) \]

Similar arguments would show

\[ \frac{i}{\sqrt{2}} \Psi_{2p+1} + \frac{i}{\sqrt{2}} \Psi_{2p-1} = \Psi_{2p_y} \quad (\text{node at } y = 0) \]